THE SEMICLASSICAL EINSTEIN-KLEIN-GORDON SYSTEM: ASYMPTOTIC ANALYSIS OF MINKOWSKI SPACETIME

Simone Murro

Department of Mathematics
University of Genoa

Working Seminar *Mathematical Physics*

June, 2026

Einstein 1915: *Gravitational Interactions* $\longleftrightarrow$ *Lorentzian Manifolds*

$$\begin{cases} G_{ab} = \kappa T_{ab} \\ (\Box_g - m^2)\varphi = 0 \end{cases}$$

GEOMETRY $\Longleftrightarrow$ Einstein tensor $G_{ab} := Ric_{ab} + g_{ab}\,(\Lambda - \frac{1}{2}R)$
MATTER $\Longleftrightarrow$ stress-energy tensor T_{ab} of classical matter $\varphi : \mathcal{M} \to \mathbb{R}$
CONSTRAINTS: diffeomorphism invariance $\longleftrightarrow$ constracted bianchi identity

Baldeschi, Gelmini, Ruffini 1983: *Dark Matter* $\longleftrightarrow$ *classical Klein-Gordon field*

The Einstein-Klein-Gordon system is linearly stable:

Using a Taylor expansion $\quad g_{ab} = \eta_{ab} + \epsilon h_{ab} + O(\epsilon^2) \quad \varphi = 0 + \epsilon\phi + O(\epsilon^2)$

$$\begin{cases} \Box h_{ab} = 0 & \text{gravitational wave equation} \\ (\delta h)_b := \eta^{ab}\partial_a h_{ab} = 0 & \text{harmonic gauge} \\ (\rho_\Sigma h)_{ab} = \gamma \in \ker K_\Sigma^\dagger := \ker(\rho_\Sigma \delta U) & \text{initial conditions} \end{cases}$$

$$\begin{cases} (\Box_\eta - m^2)\phi = 0 & \text{Klein-Gordon equation} \\ \rho_\Sigma \phi = f & \text{initial conditions} \end{cases}$$

WHAT HAPPEN FOR A QUANTIZED KLEIN-GORDON FIELD?

Møller 1962 & Rosenfeld 1963: *Classical Gravity* $\longleftrightarrow$ *Quantum Matter*

$$\begin{cases} G_{ab} = \kappa \langle :T_{ab}: \rangle_{\omega_2} \\ (\mathcal{P} \otimes 1)\, \omega_2 = (1 \otimes \mathcal{P})\, \omega_2 = 0 \end{cases}$$

QUANTUM MATTER $\Longleftrightarrow$ expectation value of renormalized quantum field $\langle :T_{ab}: \rangle_\omega$

CONSTRAINTS (for the system to be mathematically well-defined)

(1) $(\mathcal{M}, g_{ab})$ is *globally hyperbolic*, i.e. isometric to $(\mathrm{R} \times \Sigma, -\beta^2 dt^2 + \gamma_t)$

- (i) $\beta \in C^\infty(\mathcal{M}, (0, \infty))$
- (ii) γ_t is a Riemannian metric on Σ depending smoothly on $t \in \mathrm{R}$
- (iii) all sets $\{t_0\} \times \Sigma$ are Cauchy hypersurfaces in $\mathcal{M}$

(2) ω_2 is a bidistribution associated to an *Hadamard state*, i.e.

- (i) $\omega_2(f, f') - \omega_2(f', f) = i\mathcal{G}_\mathcal{P}(f, f')$ where $\ker \mathcal{P} \overset{\mathcal{G}_\mathcal{P}}{\simeq} \mathrm{C}^\infty_\mathrm{c}(\mathcal{M})$
- (ii) $|\mathcal{G}_\mathcal{P}(f, f')|^2 \leq 4\omega_2(f, f)\omega_2(f', f')$
- (iii) $\omega_2(f, f) > 0$ for all $f \notin \mathrm{ran}(\mathcal{P})$
- (iv) $\mathrm{WF}(\omega_2) = \{(x, k_x; y, -k_y) \in (T^*\mathcal{M} \times T^*\mathcal{M}) \setminus \{0\} \mid (x, k_x) \sim_{||} (y, k_y), k_x \triangleright 0\}$

GOAL: STUDY (IN)STABILITY OF THE EINSTEIN-KLEIN-GORDON SYSTEM LINEARIZED AROUND MINKOWSKI - POINCARÉ VACUUM

(I) **The Quantization of a Klein-Gordon Field**

(II) **The Formulation of the Cauchy Problem**

  (i) the forcing problem for the linearized semiclassical system
  (ii) decoupling the system via the Møller operator
  (iii) tensor decomposition in the harmonic gauge

(III) **Analysis of the Nonlocal Semiclassical System**

  (i) the well-posedness of the forcing problem
  (ii) asymptotic analysis of the solutions

(IV) **Summary and Future Outlook**

Based on

"The Semiclassical Einstein-Klein-Gordon System" – arXiv:2604.01047

S. Galanda (York) P. Meda (Trento) N. Pinamonti (Genoa) G. Schmid (Regensburg)

ALGEBRAIC APPROACH TO QUANTUM FIELD THEORY

(1) *algebra of observables* $\mathcal{A}$ encoding causal structure, dynamics, and CCR

(2) *Hadamard state* $\omega\colon \mathcal{A} \to \mathbb{C}$ which is positive, normalized and Hadamard

(i) $\omega_2(f, f') - \omega_2(f', f) = i\mathcal{G}_{\mathcal{P}}(f, f')$ $\quad\longleftrightarrow\quad$ CCR

(ii) $|\mathcal{G}_{\mathcal{P}}(f, f')|^2 \leq 4\omega_2(f, f)\omega_2(f', f')$

(iii) $\omega_2(f, f) > 0$ for all $f \notin \mathrm{ran}(\mathcal{P})$ $\quad\longleftrightarrow\quad$ positivity

(iv) $\mathrm{WF}(\omega_2) = \{(x, k_x; y, -k_y) \in (T^*\mathcal{M} \times T^*\mathcal{M}) \setminus \{0\} \mid (x, k_x) \sim_{\parallel} (y, k_y), k_x \triangleright 0\}$

Gérard and Wrochna 2012: Hadamard state $\iff b \in \Psi^1(\Sigma)$ such that

$$\mathcal{P} - (\partial_t + ib)(\partial_t - ib) = r^{\pm}_{-\infty} \in C_b^{-\infty}(\mathbb{R}, \Psi^{-\infty}(\Sigma_0))$$

For ultrastatic spacetime, i.e. $g_{ab} = -dt^2 + \gamma_{ij}$ and (Σ, γ_{ij}) is complete, being Hadamard is equivalent to solve the initial value problem for $\omega_2 \in D'(\mathcal{M} \times \mathcal{M})$

$$\begin{cases} (\mathcal{P} \otimes 1)\,\omega_2 = 0 \\ (1 \otimes \mathcal{P})\,\omega_2 = 0 \\ \rho_\Sigma \omega_2 = \frac{1}{2}\begin{pmatrix} \sqrt{-\Delta_{\gamma_{ij}} + m^2)}^{-1} & i\delta \\ -i\delta & \sqrt{-\Delta_{\gamma_{ij}} + m^2} \end{pmatrix} \end{cases}$$

Given $\omega : \mathcal{A} \to \mathbb{C} \overset{\text{GNS}}{\Longrightarrow} \pi : \mathcal{A} \to \mathsf{L}(\mathcal{H})$ operator-valued distributions but for $\phi\colon \mathcal{M} \to \mathbb{R}$

$$
\begin{aligned}
T_{ab} =& \frac{1}{2}\nabla_a\nabla_b\phi^2 - \phi\nabla_a\nabla_b\phi - \frac{1}{2}g_{ab}\phi\Box_g\phi - \frac{1}{2}g_{ab}\left(\frac{1}{2}\Box_g\phi^2 + m^2\phi^2\right) \\
&+ \xi\left(G_{ab} - \nabla_a\nabla_b + g_{ab}\Box_g\right)\phi^2
\end{aligned}
$$

involves pointwise products of field distributions $\Longrightarrow$ general ill-defined

Brunetti, Fredenhagen and Kohler 1996: *Hadamard parametrix* $\mathcal{H} \in D'(\mathcal{M} \times \mathcal{M})$

$$\omega_2 - \mathcal{H} \in C^\infty(\mathcal{M} \times \mathcal{M}) \quad \Longrightarrow \text{ well-defined restriction to } \mathrm{diag}(\mathcal{M} \times \mathcal{M})$$

Hollands and Wald 2001: *axiomatic regularisation scheme* encoding

•locality • covariance • conservation • analytic scaling behaviour

$$\langle : T_{ab} : \rangle_\omega = \lim_{y \to x} \mathcal{D}_{ab}(\omega_2 - \mathcal{H}) + \frac{A}{3}g_{ab} + \alpha_1 m^4 g_{ab} + \alpha_2 m^2 G_{ab} + \alpha_3 J_{ab} + \alpha_4 I_{ab}$$

- $\mathcal{D}_{ab}$ is a bidifferential operator derived from the classical expression for T_{ab}
- A is a coefficient depending on ξ, m, $R_{acbd}R^{acbd}$, $R_{ab}R^{ab}$, R and $\Box_g R$
- $\alpha_i \in \mathbb{R}$ are *renormalisation constants*
- I_{ab} and J_{ab} geometric tensors depending on R_{ab}, $\Box_g R_{ab}$, R and $\Box_g R$

The semiclassical EKG system coupled ω_2 and g_{ab} and require two constraints

(I) ω_2 is a Hadamard state;

(II) $(\mathcal{M}, g_{ab})$ is globally hyperbolic.

PROBLEM: initial data for $\omega_2 \Leftrightarrow$ knowing g_{ab} in a neighbourhood of Σ

OUR PROPOSAL: The Cauchy problem will be a **FORCING PROBLEM**

TOY MODEL: 1^{st}-order SHS $\mathscr{S} := \partial_t - \partial_x : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$

using $\chi(t) := \begin{cases} 1 & t \geq 0 \\ 0 & t \leq -\epsilon \end{cases} \in C^\infty(\mathbb{R})$ and $(\mathscr{S} u)\chi = \mathscr{S}(\chi u) - (\partial_t \chi) u = 0$

$$\text{(IVP)} \begin{cases} \mathscr{S} u = 0 \\ u|_{\{t=0\}} = u_0 \end{cases} \quad \overset{t\geq 0}{\Longleftrightarrow} \quad \text{(FP)} \begin{cases} \mathscr{S} \tilde{v} = f \in C^\infty_{pc}(\mathcal{M}) \\ \tilde{v} \in C^\infty_{pc}(\mathcal{M}) \end{cases}$$

The solution of (IVP) and the one of (FP) agrees for $t \geq 0$

LINEARIZED SEMICLASSICAL EINSTEIN-KLEIN-GORDON SYSTEM

We use a Taylor expansion $g_{ab} = \eta_{ab} + \epsilon h_{ab} + O(\epsilon^2)$ $\omega_2 = \omega_0 + \epsilon\omega_1 + O(\epsilon^2)$

(0) (η_{ab}, ω_0) solves the semiclassical EKG system $\iff$ finding α_1 s.t.

$$\langle :T_{ab}[\eta_{ab}]: \rangle_{\omega_0} = \lim_{y\to x} \mathcal{D}_{ab}\big(\omega_0 - \mathcal{H}\big) + \alpha_1 m^4 \eta_{ab} = 0$$

(1) the Cauchy Problem for the linearized semiclassical EKG system is

$$\begin{cases} G^{(1)}_{ab}[h_{ab}] = \kappa \frac{d}{d\lambda}\Big(\langle :T_{ab}[\eta_{ab} + \lambda h_{ab}]: \rangle_{[\omega_0 + \lambda\omega_1]}\Big)\Big|_{\lambda=0} + S_{ab} \in C^\infty_{pc}(\mathcal{M}\times\mathcal{M}) \\ \frac{d}{d\lambda}\Big((\mathcal{P}_{[\eta_{ab}+\lambda h_{ab}]} \otimes 1)(\omega_0 + \lambda\omega_1)\Big)\Big|_{\lambda=0} = \Xi \in \Gamma_{pc}(\otimes^2_s T^*\mathcal{M}) \\ \frac{d}{d\lambda}\Big((1 \otimes \mathcal{P}_{[\eta_{ab}+\lambda h_{ab}]})(\omega_0 + \lambda\omega_1)\Big)\Big|_{\lambda=0} = \Xi \in \Gamma_{pc}(\otimes^2_s T^*\mathcal{M}) \\ \omega \in D'_{pc}(\mathcal{M}\times\mathcal{M}) \\ h_{ab} \in \Gamma_{pc}(\otimes^2_s T^*\mathcal{M}) \end{cases}$$

PROBLEM: The linearized semiclassical EKG system is not decoupled

Since $g_{ab} \simeq \eta_{ab} + \epsilon h_{ab}$, the algebras of observables are $*$-isomorphic

$$\begin{array}{ccc} \mathcal{R} : \mathcal{A}_{g_{ab}} & \xrightarrow{\simeq} & \mathcal{A}_{\eta_{ab}} \\ & \omega_{g_{ab}} := \mathcal{R}^* \omega_{\eta_{ab}} \searrow & \downarrow \omega_{\eta_{ab}} = \omega_0 + \tilde{\omega} \\ & & \mathbb{C} \end{array}$$

and the dynamics are intertwined $\qquad \mathcal{P}_{[g_{ab}]}\, \omega_{g_{ab}} = \mathcal{P}_{[g_{ab}]} \mathcal{R}^*\, \omega_{\eta_{ab}} = \mathcal{P}_{[\tilde{\eta}_{ab}]}\, \omega_{\eta_{ab}}$

So we can decoupled linearized EKG equations

$$\begin{cases} G^{(1)}_{ab}[h_{ab}] = \kappa \langle (:\mathcal{R}(T_{ab}[g_{ab}]):\rangle^{(1)}_{[\omega_{\eta_{ab}}]} = S_{ab} \\ h_{ab} \in \Gamma_{pc}(\otimes^2_s T^*\mathcal{M}) \end{cases} \quad \Nwarrow$$

$$\begin{cases} (\mathcal{P}_{[\eta_{ab}]} \otimes 1)\, \omega_{\eta_{ab}} = \Xi' \\ (1 \otimes \mathcal{P}_{[\eta_{ab}]})\, \omega_{\eta_{ab}} = \Xi' \\ \omega \in D'_{pc}(\mathcal{M} \times \mathcal{M}) \end{cases}$$

nonlocal operator

$\Longleftarrow$ Cauchy problem under control

DEFINITION: Cauchy problem for the *linearized semiclassical Einstein equations*

$$\begin{cases} G^{(1)}_{ab}[h_{ab}] + \kappa \langle N_{ab}[h_{ab}] \rangle_{\omega_0} - \kappa \langle :T_{ab}[h_{ab}]: \rangle_{\omega_0} = S'_{ab} := S_{ab} - \kappa \langle :T_{ab}[\eta_{ab}]: \rangle_{\tilde{\omega}} \\ h_{ab} \in \Gamma_{pc}(\otimes^2_s T^*\mathcal{M}) \end{cases}$$

The semiclassical Einstein equations inherit a *linear gauge symmetry*

$$h_{ab} \mapsto \tilde{h}_{ab} := h_{ab} + \partial_a X_b + \partial_b X_a - \eta_{ab}\partial^c X_c$$

THEOREM: For any $h_{ab} \in \Gamma_{\mathrm{pc}}(\otimes_s^2 T^*\mathbb{R}^n)$ exists unique $X \in C^\infty_{\mathrm{pc}}(T^*\mathbb{R}^n)$

$$\partial^a \tilde{h}_{ab} = 0 \quad \textit{(de Donder gauge)} \qquad \tilde{h}_{ab} = \tilde{h}^{\mathrm{S}}_{ab} + \tilde{h}^{\mathrm{TT}}_{ab} \quad \textit{(tensor decomposition)}$$

where $\tilde{h}^{\mathrm{TT}}_{ab} \in \Gamma_{\mathrm{pc}}(\otimes_s^2 T^*\mathbb{R}^n)$ with $\partial^a \tilde{h}^{\mathrm{TT}}_{ab} = 0$ and $\eta^{ab}\tilde{h}^{\mathrm{TT}}_{ab} = 0$

$$\tilde{h}^{\mathrm{S}}_{ab} = \frac{1}{n-1}\left(\eta_{ab}\tilde{h}^c_c - \partial_a\partial_b \mathrm{G}_{\mathrm{ret}}(h)\right)$$

Notice that $\partial^a \tilde{h}_{ab} = 0 \Longleftrightarrow \partial^a h_{ab} = \Box X_b \implies$ complete gauge fixing!

Forcing Problem for the linearized semiclassical Einstein equations

$$\begin{cases} -\frac{1}{2\kappa}\Box h^{\mathrm{S}}_{ab} + \frac{1}{4}\mathcal{K}_0[\mathcal{S}h^{\mathrm{S}}_{ab}] - \frac{m^4}{64\pi^2}h^{\mathrm{S}}_{ab} + \frac{1}{2}\tilde{\alpha}^{\mathrm{S}}_2 m^2 \Box h^{\mathrm{S}}_{ab} - \tilde{\alpha}^{\mathrm{S}}_3 \Box\Box h^{\mathrm{S}}_{ab} = S^{\mathsf{S}}_{ab} \in \Gamma_c(\otimes_s^2 T^*\mathcal{M}) \\ \partial^a h^{\mathrm{S}}_{ab} = 0 \end{cases}$$

$$\begin{cases} -\frac{1}{2\kappa}\Box h^{\mathrm{TT}}_{ab} - \frac{1}{2}\mathcal{K}_0[\mathcal{T}h^{\mathrm{TT}}_{ab}] + \frac{1}{2}\tilde{\alpha}^{\mathrm{TT}}_2 m^2 \Box h^{\mathrm{TT}}_{ab} - \tilde{\alpha}^{\mathrm{TT}}_4 \Box\Box h^{\mathrm{TT}}_{ab} = S^{\mathrm{TT}}_{ab} \in \Gamma_c(\otimes_s^2 T^*\mathcal{M}) \\ \partial^a h^{\mathrm{TT}}_{ab} = 0 \end{cases}$$

where $\mathcal{K}_0(-) = \dfrac{1}{16\pi^2}\Box \displaystyle\int_{m^2}^\infty \rho(M)\mathrm{G}^M_{\mathrm{ret}}(-)\,dM \qquad \mathcal{S},\mathcal{T} \in \mathrm{Diff}^2_{\mathsf{hyp}}(\otimes_s^2 T^*\mathcal{M})$

TOY MODEL for the Semiclassical Einstein Equations with $\phi \in C^\infty_{\mathrm{pc}}(\mathcal{M})$

$$\mathcal{K}_0[(\Box - a_1)(\Box - a_2)\phi] + \sum_{j=0}^{2} b_j \Box^j \phi = S$$

PROPOSITION $\iota_{\mathcal{M}} : \mathcal{M} \hookrightarrow \mathcal{M}_5 := \mathcal{M} \times \mathbb{R}$ and we consider

$$\tilde{\mathrm{G}}^{4m^2}_{\mathrm{ret}} : C^\infty_{\mathrm{pc}}(\mathcal{M}) \otimes L^2(\mathbb{R}) \to C^\infty_{\mathrm{pc}}(\mathcal{M}) \otimes C^0(\mathbb{R})$$

Then, for every $\phi \in C^\infty_{\mathrm{pc}}(\mathcal{M})$, we have that

$$\mathcal{K}_0(\phi) := \frac{1}{16\pi^2}\Box \int_{m^2}^{\infty} \rho(M) \mathrm{G}^M_{\mathrm{ret}}(\phi)\, dM = \iota^*_{\mathcal{M}} \tilde{\mathrm{G}}^{4m^2}_{\mathrm{ret}}(\Box\phi \otimes \check{\rho}) = \tilde{\mathrm{G}}^{4m^2}_{\mathrm{ret}}(\Box\phi \otimes \check{\rho})|_{z=0}$$

where $\check{\rho}(z) = \int_{\mathbb{R}} \rho(p_z^2 + 4m^2)|p_z| e^{\mathrm{i}p_z z}\, dp_z$.

If ς is positive, square-integrable on $[4m^2, \infty)$ and Hölder continuous, then

$$C^\infty_{pc}(\mathcal{M}) \ni f \mapsto \mathcal{G}_\varsigma(f) := \iota^*_{\mathcal{M}} \tilde{\mathrm{G}}^{4m^2}_{\mathrm{ret}}(f \otimes \check{\varsigma}) \quad \text{is invertible}$$

$$\textbf{TOY MODEL} \Leftrightarrow \underbrace{\iota^*_{\mathcal{M}} \tilde{\mathrm{G}}^{4m^2}_{\mathrm{ret}}}_{\text{order -2}} \Big(\underbrace{\Box(\Box - a_1)(\Box - a_2)}_{\text{order 6}} \phi \otimes \check{\rho} \Big) + \underbrace{\sum_{j=0}^{2} b_j \Box^j \phi}_{\text{order 4}} = S$$

NOTICE: inverting $\iota^*_{\mathcal{M}} \tilde{\mathrm{G}}^{4m^2}_{\mathrm{ret}}$ does not help ad this stage

We start with rewriting the TOY MODEL in the following form

$$\iota^*_{\mathcal{M}} \tilde{\mathsf{G}}^{4m^2}_{\mathrm{ret}} \left(\Box(\Box - a_1)(\Box - a_2)\phi \otimes \check{\rho} + \sum_{j=0}^{2} (\tilde{\Box} - 4m^2)(b_j \Box^j \phi \otimes \mathfrak{h}_j) \right) = S$$

where $\tilde{\Box}$ is the d'Alembert operator on $\mathcal{M}_5$, and with a direct computation

$$\iota^*_{\mathcal{M}} \tilde{\mathsf{G}}^{4m^2}_{\mathrm{ret}} \left(\Box(\Box - a_1)(\Box - a_2)\phi \otimes \check{\rho} + \sum_{j=0}^{2} ((\Box - 4m^2) b_j \Box^j \phi \otimes \mathfrak{h}_j) + \sum_{j=0}^{2} b_j \Box^j \phi \otimes \partial_z^2 \mathfrak{h}_j \right) = S$$

If $\mathfrak{h}_j$ satisfy the following ODEs for some constants β_i and with $\mathfrak{h}_j(0) = 1$

$$\begin{cases} b_2 \partial_z^2 \mathfrak{h}_2 - 4m^2 b_2 \mathfrak{h}_2 + b_1 \mathfrak{h}_1 - (a_1 + a_2)\check{\rho} = \beta_2(\check{\rho} + b_2 \mathfrak{h}_2) \\ b_1 \partial_z^2 \mathfrak{h}_1 - 4m^2 b_1 \mathfrak{h}_1 + b_0 \mathfrak{h}_0 + (a_1 a_2)\check{\rho} = \beta_1(\check{\rho} + b_2 \mathfrak{h}_2) \\ b_0 \partial_z^2 \mathfrak{h}_0 - 4m^2 b_0 \mathfrak{h}_0 = \beta_0(\check{\rho} + b_2 \mathfrak{h}_2). \end{cases} \tag{1}$$

then the TOY MODEL simplifies as

$$\iota_{\mathcal{M}} \tilde{\mathsf{G}}_{\mathrm{ret}}((\Box^3 \phi + \beta_2 \Box^2 \phi + \beta_1 \Box \phi + \beta_0 \phi) \otimes (\check{\rho} + b_2 \mathfrak{h}_2)) = S$$

if $\check{\rho} + b_2 \mathfrak{h}_2$ is positive and sufficiently regular, then the TOY MODEL reads

$$\Box^3 \phi + \beta_2 \Box^2 \phi + \beta_1 \Box \phi + \beta_0 \phi = S_N$$

PROPOSITION: $\forall\ a_1, a_2 \leq 4m^2$ and $\forall\ \tilde{b}_0, \tilde{b}_1, \tilde{b}_2 \in \mathbb{R}$, there exist $b_0, b_1 \in \mathbb{R}$ s.t.

$$\iota^*_{\mathcal{M}} \tilde{\mathsf{G}}^{4m^2}_{\mathrm{ret}} (\Box(\Box - a_1)(\Box - a_2)\phi \otimes \check{\rho}) + \sum_{j=0}^{2} \tilde{b}_j \Box^j \phi = S$$

$$\Updownarrow$$

$$\underbrace{\Box^3\phi + \beta_2\Box^2\phi + \beta_1\Box\phi + \beta_0\phi}_{L\phi} + \mathcal{G}_\varsigma^{-1}((\tilde{b}_1 - b_1)\Box\phi + (\tilde{b}_0 - b_0)\phi) = \mathcal{G}_\varsigma^{-1}(S)$$

$$\Updownarrow \mathsf{G}_{L,\mathrm{ret}}$$

$$\phi + W_{\mathrm{ret}}(\phi) = \mathsf{S}$$

WE SOLVE PERTURBATIVELY

$$\phi_0 := \mathsf{S}, \qquad \phi_n := W_{\mathrm{ret}}(\phi_{n-1}) \quad \text{for} \quad n > 0\,.$$

THEOREM: The series defined by

$$\phi = \sum_{n\geq 0} (-1)^n \phi_n = \sum_{n\geq 1} (-1)^n W^n_{\mathrm{ret}}(\mathsf{S})$$

converges absolutely in $C^\infty_{\mathrm{pc}}(\mathcal{M})$

TOY MODEL $\quad \mathcal{K}_0[(\Box - a_1)(\Box - a_2)\phi] + \sum_{j=0}^{2} b_j \Box^j \phi = S\,,$

$$\Updownarrow\; w^2 := s^2 + \mathrm{p}^2$$

$\mathscr{F}$**ourier**–$\mathscr{L}$**aplace** $\quad Q(w^2)(\mathscr{L}\mathscr{F}\phi(s,\mathrm{p})) = \mathscr{L}\mathscr{F}S(s,\mathrm{p})$

Hence, the solution of the **TOY MODEL** can be written as

$$\mathscr{L}\mathscr{F}(\phi)(s,\mathrm{p}) = \frac{1}{Q(w^2)}\hat{S}(s,\mathrm{p})$$

Using Stieltjes-Perron inversion formula, we get

$$\phi = \sum_{\gamma\in\mathcal{Z}} \frac{1}{Q'(-\gamma)} \mathrm{G}^{\gamma}_{\mathrm{ret}}(S) + \int_{4m^2}^{\infty} \vartheta(M)\mathrm{G}^{M}_{\mathrm{ret}}(S)\, dM$$

where $\mathcal{Z}$ is the set of zeros of Q in $\mathbb{C}\setminus(-\infty,-4m^2)$.

THEOREM: (1) If $\mathcal{Z}\subset(-4m^2,0) \iff \phi$ decays at least as $t^{-3/2}$ for large time

(2) If $\mathcal{Z}\subset(0,\infty) \iff \phi$ is exponentially growing

(S) b_2 is a free parameter which depends $\tilde{\alpha}_3^{\mathrm{S}}$

$$a_1 = a_2 = \frac{2m^2}{6\xi - 1}, \quad b_0 = -\alpha_1^{\mathrm{S}} \frac{4m^4}{6\left(\frac{1}{6} - \xi\right)^2}, \quad b_1 = -\frac{2}{\kappa}\frac{1}{6\left(\frac{1}{6} - \xi\right)^2}$$

(TT) b_2 is a free parameter proportional to $\tilde{\alpha}_4^{\mathrm{TT}}$

$$a_1 = a_2 = 4m^2, \qquad b_0 = 0, \qquad b_1 = \frac{60}{\kappa}$$

MAIN RESULT: For any compacty supported Cauchy data, the solution of the semiclassical Einstein equations grows exponentially in time but bounded by

$$\tilde{h}_{ab} \leq e^{Ht}\eta_{ab}$$

with H an effective Hubble constant depending on m of the Klein-Gordon field

$$\Downarrow$$

H match experiment if $m \simeq 7.8 \times 10^{-3}$ eV which is compatible with axions-particles

CONGECTURE: The de Sitter universe $(\mathbb{R}^4, g_{ab} := e^{Ht}\eta_{ab})$ is linearly stable

THANKS for your attention!