# ON THE CONSTRUCTION OF HADAMARD STATES FOR THE LINEARIZED EINSTEIN EQUATIONS

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Working Seminar *Mathematical Physics*

May, 2024

DIMA
UniGe
MUR 2023-2027

# THE LINEARIZED EINSTEIN EQUATIONS

Let $(\mathsf{M}, g)$ be a **globally hyperbolic manifold** satistying

$$\mathsf{Ric}_g - \frac{1}{2}\mathsf{Scal}_g\, g + \Lambda\, g = 0 \tag{1}$$

The **linearized Einstein's equations** can be described by

$$\mathsf{P} \colon \Gamma(\mathsf{V}_2) \to \Gamma(\mathsf{V}_2), \qquad \mathsf{P} := \mathsf{D}_2 - \mathsf{K}\mathsf{K}^\star\,,$$

where

- $\mathsf{D}_2 := -\Box_2 + 2\mathrm{Riem}_g$ is a normally hyperbolic operator
- $\mathsf{K} := \mathrm{I} \circ \mathrm{d}$ is the linear counterpart of the diffeomorphism invariance of (1)
  - → $\mathrm{I} := \mathbb{1} - \frac{1}{2}\mathrm{tr}_g(\cdot)g$ denotes the **trace-reversal**;
  - → $(\mathrm{d}\omega)_{\alpha\beta} = \frac{1}{2}(\nabla_\alpha\omega_\beta + \nabla_\beta\omega_\alpha)$ is the **symmetrised gradient**;
- $\mathsf{K}^\star = \delta$ that is the **divergence** of $u \in \Gamma(\mathsf{V}_2)$, i.e. $(\delta u)_\mu = -2\nabla^\lambda u_{\lambda\mu}$ .

# THE FIRST DIFFULTIES

1. $P$ is not hyperbolic since it holds $PK = 0$

   $\Rightarrow$ linearized gravity should be treat as gauge theory

$$\frac{\ker_{sc} \mathrm{P}}{\mathrm{ran} K} \simeq \frac{\ker \mathrm{D}_2 \cap \ker \mathrm{K}^\star}{\mathrm{K}(\ker \mathrm{D}_1)} \simeq \frac{\ker \mathrm{K}_\Sigma^\dagger}{\mathrm{ran}\, \mathrm{K}_\Sigma}$$

where

$\rightarrow$ $\mathrm{K}_\Sigma^\dagger := \rho_0 K^* U_1$ is the gauge condition at the level of initial data ;

$\rightarrow$ $\mathrm{K}_\Sigma := \rho_1 K U_0$ is the gauge symmetry at the level of the initial data.

2. $\mathrm{D}_2 u = 0$ and $\mathrm{K}^\star u = 0$ is an overdetermined system

   $\Rightarrow$ we find solutions if $(M, g)$ solves the Einstein's equations since

$$\mathrm{D}_2 \mathrm{K} = \mathrm{K} \mathrm{D}_1 \qquad \text{for} \qquad \mathrm{D}_1 = -\Box_1 - \Lambda$$

3. $P_2$ is self-adjoint w.r.t. the indefinite inner product

$$(\cdot\,,\,\cdot)_{\mathrm{V}_2,\mathrm{I}} := 2 \int_{\mathrm{M}} \otimes_s^2 g^{-1}(\overline{\cdot}, \mathrm{I}\cdot)\,\mathrm{vol}$$

# QUANTIZATIONS IN TWO STEPS

**Step 1**: From the classical phase space

$$\frac{\ker_{sc} \mathsf{P}}{\mathrm{ran}_{sc} \mathsf{K}} \xrightarrow[\simeq]{\mathsf{G_1}} \frac{\ker_c \mathsf{K}^\star}{\mathrm{ran}_c \mathsf{P}} =: \mathcal{V}_{\mathrm{P}} \quad \mathrm{q}_{2,\mathrm{I}}(\cdot|\cdot) := \mathrm{i}(\cdot|\mathsf{G}_2\cdot)_{\mathsf{V}_2,\mathrm{I}}$$

we construct an abstract $*$-algebra $\mathrm{CCR}$

generators: $\Phi(v) \qquad \Phi^*(v) \qquad \mathbb{1}$

CCR relations:

$$[\Phi(v), \Phi(w)] = [\Phi^*(v), \Phi^*(w)] = 0$$
$$[\Phi(v), \Phi^*(w)] = \mathrm{q}_{2,\mathrm{I}}(v, w)\mathbb{1}$$

**Step 2**: Construct an **Hadamard states** $\omega : \mathrm{CCR} \to \mathbb{C}$ defined by

covariances: $\Lambda^+(v, w) := \omega(\Phi(v)\Phi^*(w)) \qquad \Lambda^-(v, w) := \omega(\Phi^*(w)\Phi(v))$

Hadamard conditions: $\mathrm{WF}'(\Lambda^\pm) \subset \mathcal{N}^\pm \times \mathcal{N}^\pm$

$$\mathcal{N}^\pm := \{(p, \xi) \in \mathrm{T}^*\mathrm{M} \mid g^{-1}(\xi, \xi) = 0 \text{ and } \pm\xi(v) > 0 \ \forall v \in \mathrm{T}_p\mathrm{M} \text{ fut.dir.}\}$$

# CHARACTERIZATION OF HADAMARD STATES

We work with initial data

$$(\mathcal{V}_{\mathrm{P}}, \mathrm{q}_{2,\mathrm{I}}) \xrightarrow[\simeq]{[\rho \mathsf{G}_2]} \left(\mathcal{V}_\Sigma := \frac{\ker \mathsf{K}_\Sigma^\dagger}{\operatorname{ran} \mathsf{K}_\Sigma}, \mathrm{q}_{2,\mathrm{I},\Sigma}(\cdot|\cdot) = \mathrm{i}(\cdot|\mathsf{G}_{2,\Sigma}\cdot)_{\mathsf{V}_{\rho_2|\mathrm{I}}}\right)$$

PROPOSITION [Gérard-M.-Wrochna]: let $c_2^\pm : \Gamma(\mathsf{V}_{\rho_2}) \to \Gamma(\mathsf{V}_{\rho_2})$ be

(i) $c_2^+ + c_2^- = \mathbb{1}$ ;

(ii) $(c_2^\pm)^\dagger = c_2^\pm$ (w.r.t. $q_{2,\mathrm{I},\Sigma}$);

(iii) $\mathrm{q}_{2,\Sigma}(\mathfrak{f}|c_2^\pm \mathfrak{f}) \geqslant 0 \quad \forall \mathfrak{f} \in \ker(\mathsf{K}_\Sigma^\dagger)$ .

(iv) $c_2^\pm \mathsf{K}_\Sigma = \mathsf{K}_\Sigma c_1^\pm$ for some $c_1^\pm$

(v) $\mathrm{WF}'(U_1 c^\pm) \subset (\mathcal{N}^\pm \cup F) \times \mathrm{T}^*\Sigma$ for some $F \subset \mathrm{T}^*\mathrm{M}$ s.t. $F \cap \mathcal{N} = \emptyset$

Then $\Lambda^\pm([s],[t]) := (s, \lambda^\pm t)_{\mathsf{V}_1}$ where $\lambda^\pm := \pm \mathrm{i}(\rho_2 \mathsf{G}_2)^* \mathsf{G}_{2,\Sigma} c_2^\pm (\rho_2 \mathsf{G}_2)$ are pseudo-covariances for a quasifree Hadamard state $\omega : \mathrm{CCR} \to \mathbb{C}$.

HOW CAN WE CONSTRUCT THESE OPERATORS?

# A TOY MODEL FROM ELLIPTIC THEORY: CALDERÓN PROJECTORS

Consider a Riemannian manifold $(\Sigma, h)$ and define

$$\Omega = \mathbb{R} \times \Sigma\,, \qquad \Omega^\pm = \Omega \cap \mathbb{R}^\pm\,, \qquad \text{with metric } g = ds^2 + h\,.$$

Let $\overline{\mathcal{D}'}(\Omega^\pm)$ be the space of distribution in $\mathcal{D}'(\Omega)$ restricted to $\Omega^\pm$ and set

$$N^\pm := \{u \in \overline{\mathcal{D}'}(\Omega^\pm) \,|\, \tilde{P}u := (-\partial_s^2 \underbrace{-\Delta_h + m^2}_{\varepsilon^2})u = 0\}\,.$$

For any $u \in N^\pm$, the traces $\varrho^\pm u = \begin{pmatrix} u|_{\partial\Omega^\pm} \\ -\partial_s u|_{\partial\Omega^\pm} \end{pmatrix}$ are well-defined and

Calderón projectors $c^\pm$ are defined to be projectors onto

$$Z^\pm := \varrho^\pm N^\pm := \{f \in \mathcal{D}'(\Sigma; \mathbb{C}^2) \,:\, f = \varrho^\pm u\,, \text{for } u \in N^\pm\}$$

Using that $u = e^{\mp\varepsilon s} v$ for $v \in \mathcal{D}'(\Sigma)$ and $Z^\pm$ are linearly independet, we have

$$\varrho^\pm u = \begin{pmatrix} u|_{\partial\Omega^\pm} \\ -\partial_s u|_{\partial\Omega^\pm} \end{pmatrix} = \begin{pmatrix} v \\ \pm\varepsilon v \end{pmatrix} \qquad \text{and} \qquad c^\pm = \frac{1}{2}\begin{pmatrix} 1 & \pm\epsilon^{-1} \\ \pm\epsilon & 1 \end{pmatrix}$$

The Caldéron projectors are associated to the ground states!

**Additional difficulties:** $m = 0 \Longrightarrow$ infrared divergences!

# OUTLINE

(I) Wick rotation in analytic spacetimes

(II) Construction of Calderón projectors

(III) Boundary conditions for linearized gravity

**Based on**

"*Wick rotation of linearized gravity in Gaussian time and Calderón projectors*"
( with C. Gérard and M. Wrochna )

"*On Boundary Conditions for Linearised Einstein's Equations*"
( with M. Capofferi and G. Schmid)

# GEOMETRIC SETTING

As always $(\mathrm{M}, g)$ is a globally hyperbolic spacetime

- To deal with the overdetermined Cauchy problem, we require

(1) $g$ solves the Einstein's equations

- To have a good pseudodifferential calculus, we assume:

(2) $(\mathrm{M}, g)$ to be of *bounded geometry* near a Cauchy surface $\Sigma$

- To implement the Wick rotation, we use Gaussian normal coordinate near $\Sigma$

$$[-\delta, \delta] \times \Sigma \qquad g = -dt^2 + h_t$$

and suppose: (3) the map $t \mapsto h_t$ is *real analytic* for $0 < \delta \ll 1$

THEOREM [C.Gérard, S.M., M.Wrochna]

Let $(\Sigma, h)$ be a Riemannian manifold of analytic bounded geometry s.t. the Cauchy data for the Einstein's equations satisfying the usual constraints.

$\Longrightarrow$ **there exists a metric $g$ such that $(\mathrm{M}, g)$ satisfies (1)-(3)**

# REDUCED SETTING AND WICK ROTATION

Now we fix Gaussian normal coordinates

$$M_\delta := I_\delta \times \Sigma \qquad g = -dt^2 + h_t$$

and use the decompositions

$$\omega = \omega_t dt + \omega_\Sigma \qquad u = u_{tt} dt \otimes_s dt + u_{t\Sigma} \otimes_s dt + u_{\Sigma\Sigma}$$

We identify tensors on different $\Sigma_t$ on the same $\Sigma_0$ by parallel transport along $\partial_t$ and the differential operators $\mathsf{D}_j$ rewrite as

$$\mathsf{D}_j = \partial_t^2 + a_j \qquad \text{with principal symbol } \sigma_{\mathrm{pr}}(a_j)(k) = h_t^{-1}(k,k)\mathbb{1}$$

WICK ROTATION:

Since $t \mapsto h_t$ is real analytic, all the operators extend holomorphically in $\mathbb{C} \times \Sigma$

$$\mathsf{D}_j = \partial_t^2 + a_j \quad \text{(hyperbolic)} \quad \xrightarrow{t=is} \quad \tilde{\mathsf{D}}_j = -\partial_s^2 + a_j(is) \quad \text{(elliptic)}$$

$$\mathsf{D}_2\mathsf{K} = \mathsf{K}\mathsf{D}_1 \quad \Rightarrow \quad \tilde{\mathsf{D}}_2\tilde{\mathsf{K}} = \tilde{\mathsf{K}}\tilde{\mathsf{D}}_1$$

# DIRICHLET REALIZATION

To construct Calderón proj. we need a boundary condition on $\Omega := [-T, T] \times \Sigma$

$\Longrightarrow$ **Dirichlet boundary conditions**

Let $H_0^1(\Omega; \tilde{V})$ be the closure of $C_c^\infty(\Omega, \tilde{V})$ for the norm

$$\|u\|^2_{H^1(\Omega;\tilde{V})} = \int_\Omega \left( (\partial_s u|\partial_s u)_{\tilde{V}} + (u| - \Delta_{\tilde{h}_0} u)_{\tilde{V}} + (u|u)_{\tilde{V}} \right) |\tilde{h}_0|^{\frac{1}{2}} dt d\mathrm{x}.$$

and consider the sesquilinear form

$$Q_\Omega(v, u) := (v|\tilde{\mathrm{D}}u)_{\tilde{V}(\Omega)}, \text{ with domain } \operatorname{Dom} Q_\Omega = C_c^\infty(\Omega; \tilde{V}).$$

Then we have:

- $Q_\Omega$ and $Q_\Omega^*$ are closeable on $L^2(\Omega; \tilde{V})$;
- their closures $\overline{Q_\Omega}$, $\overline{Q_\Omega^*}$ are sectorial with domain $H_0^1(\Omega; \tilde{V})$;
- $\tilde{\mathrm{D}}_\Omega$, $\tilde{\mathrm{D}}_\Omega^*$ associated to $\overline{Q_\Omega}$, $\overline{Q_\Omega^*}$ satisfy $0 \in \mathrm{rs}(\tilde{\mathrm{D}}_\Omega)$ $0 \in \mathrm{rs}(\tilde{\mathrm{D}}_\Omega^*)$
- $\tilde{\mathrm{D}}_\Omega^*$ is the adjoint of $\tilde{\mathrm{D}}_\Omega$.

DEFINITION: $\tilde{\mathrm{D}}_\Omega$ is called **Dirichlet realization** of $\tilde{\mathrm{D}}$

# CALDERÓN PROJECTORS WITH DIRICHLET BOUNDARY CONDITIONS

**DEFINITION:** The **Calderón projectors** for the Dirichlet realization $\tilde{\mathrm{D}}_\Omega$ of $\tilde{\mathrm{D}}$ are

$$\tilde{c}^{\pm} := \mp \tilde{\varrho}^{\pm} \tilde{\mathrm{D}}_\Omega^{-1} \tilde{\varrho}^* \tilde{\sigma}$$

where

- $\tilde{\varrho}^{\pm} u = \begin{pmatrix} u(0^{\pm}) \\ -\partial_s u(0^{\pm}) \end{pmatrix}$
- $\tilde{\sigma} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
- $\tilde{\varrho}^*$ is the adjoint of $\tilde{\varrho}$
- $\tilde{\mathrm{D}}_\Omega^{-1}$ is the inverse of $\tilde{\mathrm{D}}_\Omega$

To study the microlocal properties of $\tilde{c}^{\pm}$ we need a parametrix $\tilde{\mathrm{D}}_\Omega^{(-1)}$ for $\tilde{\mathrm{D}}_\Omega$ s.t.

$$\tilde{\mathrm{D}}_\Omega^{-1} - \tilde{\mathrm{D}}_\Omega^{(-1)} \in \mathcal{W}^{-\infty}(\Omega; \tilde{V})$$

To this end we need

1) microlocal factorization for $\tilde{\mathrm{D}}$

2) a parametrix for $\tilde{\mathrm{D}}$

# MICROLOCAL FACTORIZATION OF $\tilde{\mathrm{D}}$

- $\tilde{\mathrm{D}} = -\partial_s^2 + \tilde{a}$ has the following microlocal factorization

$$\tilde{\mathrm{D}} - \tilde{r}^{\pm}_{-\infty} = (-\partial_s + \tilde{b}^{\pm})(\partial_s + \tilde{b}^{\pm})$$

(sketch of the proof)

We add to $\tilde{a}$ a smoothing operator $\tilde{r}_{-\infty} = \tilde{r}^{*}_{-\infty}$ s.t.

$$\tilde{a} + r_{-\infty} \quad \text{is } m\text{-accretive} \quad \Longrightarrow \quad \tilde{\epsilon} := (\tilde{a} + r_{-\infty})^{\frac{1}{2}}$$

The operator $\tilde{\epsilon}$ with domain $H^1(\Sigma; \tilde{V})$ is closed, elliptic, invertible and

$$\sigma_{\mathrm{pr}}(\tilde{\epsilon}) = (\sigma_{\mathrm{pr}}(\tilde{a}))^{\frac{1}{2}}$$

We add to $\tilde{\epsilon}$ a $\tilde{b}_0 \in \Psi^0(\Sigma; \tilde{V})$ s.t. $\pm\tilde{b}^{\pm} = \tilde{\epsilon} \pm \tilde{b}_0$ are $m$-accreative and

$$\partial_s \tilde{b}^{\pm}(s) - (\tilde{b}^{\pm})^2(s) + \tilde{a}(s) = \tilde{r}^{\pm}_{-\infty}(s)$$

□

# PARAMETRIX FOR $\tilde{\mathrm{D}}$

For any $\mp(s-s')\geqslant 0$ we define the operator

$$V^{\pm}(s,s') := \mathrm{Texp}\big(\textstyle\int_{s'}^{s}\tilde{b}^{\pm}(\sigma)d\sigma\big)$$

For $v\in C^\infty_{\mathrm{b}}(I;C^\infty_{\mathrm{c}}(\Sigma;\tilde{V}))$ we set

$$T^{\pm}v(s) := \pm\int_{\mathbb{R}} H(\mp(s-s'))V^{\pm}(s,s')v(s')ds',$$

where $H(t)=1_{\mathbb{R}^+}(t)$ is the Heaviside function, so that

$$(-\partial_s+\tilde{b}^{\pm})\circ T^{\pm} = T^{\pm}\circ(-\partial_s+\tilde{b}^{\pm}) = 1.$$

- $\tilde{\mathrm{D}}^{(-1)} = \Big((\tilde{b}^{+}-\tilde{b}^{-})^{-1}(T^{+}-T^{-})\Big)$ is a parametrix for $\tilde{\mathrm{D}}$

$$\tilde{\mathrm{D}}\circ\tilde{\mathrm{D}}^{(-1)} = 1 + R_{-\infty},$$

# PARAMETRIX FOR $\tilde{\mathrm{D}}_\Omega$ & MICROLOCAL ESPRESSION FOR $\tilde{c}^\pm$

To define a parametrix we need the following opertors:

- $W^\pm(s,s') = \mathrm{Texp}(-\int_{s'}^{s} \tilde{b}^\pm(\sigma)d\sigma)$, for $\pm(s-s') \geqslant 0$
- $R_{1,-\infty} = \begin{pmatrix} 0 & W^-(-T,T) \\ W^+(T,-T) & 0 \end{pmatrix}$
- $S\begin{pmatrix} v^+ \\ v^- \end{pmatrix}(s) := W^+(s,-T)v^+ + W^-(s,T)v^-$,
- $\varrho_{\partial\Omega} f := \begin{pmatrix} f(-T) \\ f(T) \end{pmatrix}$,

- $\tilde{\mathrm{D}}_\Omega^{(-1)} = \tilde{\mathrm{D}}^{(-1)} - S \circ (1 + R_{1,-\infty})^{-1} \circ \varrho_{\partial\Omega} \circ \tilde{\mathrm{D}}^{(-1)}$ is a parametrix for $\tilde{\mathrm{D}}_\Omega$ and the Calderón projectors $\tilde{c}^\pm = \mp\tilde{\varrho}^\pm \tilde{\mathrm{D}}_\Omega^{-1} \tilde{\varrho}^* \tilde{\sigma}$ can be written modulo smoothing as

$$\tilde{c}^\pm = \begin{pmatrix} \mp(\tilde{b}^+ - \tilde{b}^-)^{-1}\tilde{b}^\mp & \pm(\tilde{b}^+ - \tilde{b}^-)^{-1} \\ \mp\tilde{b}^+(\tilde{b}^+ - \tilde{b}^-)^{-1}\tilde{b}^- & \pm\tilde{b}^\pm(\tilde{b}^+ - \tilde{b}^-)^{-1} \end{pmatrix}(0) + R^\pm_{-\infty}$$

# PRO AND CONS FOR DIRICHLET BOUNDARY CONDITIONS

## PRO:

PROPOSITION [C.Gérard, S.M., M.Wrochna]

(i) $\tilde{c}_j^{\pm} : \mathcal{H}^s(\Sigma; \tilde{V} \otimes \mathbb{C}^2) \to \mathcal{H}^s(\Sigma; \tilde{V} \otimes \mathbb{C}^2)$ are bounded

(ii) $\tilde{c}_j^{+} + \tilde{c}_j^{-} = \mathbb{1}$,

(iii) $\tilde{c}_j^{\pm} = (\tilde{c}_j^{\pm})^2$

(iv) $\tilde{c}_j^{\pm} = (\tilde{c}_j^{\pm})^{\dagger}$ (w.r.t. $q_{2,\Sigma}$)

(v) $\mathrm{WF}'(U(\cdot,0)\tilde{c}^{\pm}) \subset (\mathcal{N}^{\pm} \cup \mathcal{F}) \times T^*\Sigma$ for $\mathcal{F} = \{k = 0\} \subset T^*M$.

## CONS:

PROPOSITION [C.Gérard, S.M., M.Wrochna]

(vii) $\tilde{c}_2^{\pm} \mathrm{K}_\Sigma = \mathrm{K}_\Sigma \tilde{c}_1^{\pm} \pm \mathrm{K}_{-\infty}^{\pm}$ (gauge invariance up to smoothing)

(viii) $\mathrm{q}_{2,\mathrm{I},\Sigma}(\mathfrak{f} | c_2^{\pm} + \tilde{r}_{2,-\infty}\mathfrak{f}) \geqslant 0 \quad \forall \mathfrak{f} \in \ker(\mathrm{K}_\Sigma^{\dagger})$ (positivity up to smoothing)

# SOURCE OF THE SMOOTHING OBSTRUCTION: SKETCH OF THE PROOF

(vii) Let $\kappa_2$ be such that $\tilde{\varrho}^+\kappa_2 = -\tilde{\varrho}^-$ and set for $f_i \in C^\infty_{\mathrm{c}}(\Sigma; \tilde{V}_i \otimes \mathbb{C}^2)$:

$$u_2 = -\tilde{\mathsf{D}}_{2\Omega}^{-1}\tilde{\mathsf{K}}\tilde{\varrho}_1^*\tilde{\sigma}_1 f_1, \quad v_2 = -\kappa_2\tilde{\mathsf{D}}_{2\Omega}^{-1}\tilde{\varrho}_2^*\tilde{\sigma}_2 f_2$$

Since $\tilde{\mathsf{D}}_2\tilde{\mathsf{K}} = \tilde{\mathsf{K}}\tilde{\mathsf{D}}_1$ as differential operator but $\tilde{\mathsf{K}}\,\mathrm{dom}(\tilde{\mathsf{D}}_{1,\Omega}) \not\subset \mathrm{dom}(\tilde{\mathsf{D}}_{2,\Omega})$

$$\begin{aligned}\tilde{\varrho}_2^+ u_2 =& \tilde{\varrho}_2^+\tilde{\mathsf{D}}_{2\Omega}^{-1}\tilde{\mathsf{K}}\tilde{\varrho}_1^*\tilde{\sigma}_1 f_1 = \tilde{\varrho}_2^+\tilde{\mathsf{D}}_{2\Omega}^{-1}\tilde{\mathsf{K}}\tilde{\mathsf{D}}_1\tilde{\mathsf{D}}_{1\Omega}^{-1}\tilde{\varrho}_1^*\tilde{\sigma}_1 f_1\\ =& \tilde{\varrho}_2^+\tilde{\mathsf{D}}_{2\Omega}^{-1}\tilde{\mathsf{D}}_2\tilde{\mathsf{K}}\tilde{\mathsf{D}}_{1\Omega}^{-1}\tilde{\varrho}_1^*\tilde{\sigma}_1 f_1 = \tilde{\varrho}_2^+\tilde{\mathsf{K}}\tilde{\mathsf{D}}_{1\Omega}^{-1}\tilde{\varrho}_1^*\tilde{\sigma}_1 f_1 + r^+_{-\infty} f_1\\ =& \mathsf{K}_\Sigma\rho_1^+\tilde{\mathsf{D}}_{1\Omega}^{-1}\tilde{\varrho}_1^*\tilde{\sigma}_1 f_1 + r^+_{-\infty} f_1 = \mathsf{K}_\Sigma\tilde{c}_1^+ f_1 + r^+_{-\infty} f_1\end{aligned}$$

We have $\tilde{\mathsf{D}}_2^* v_2 = 0$ in $\Omega^+$, $\tilde{\mathsf{D}}_2 u_2 = 0$ in $\Omega^+$ and the Green identity yields

$$q_{2,\mathrm{I},\Sigma}(\tilde{\varrho}_2^+ v_2|\tilde{\varrho}_2^+ u_2)_{\tilde{V}_2(\Sigma)\otimes\mathbb{C}^2} = 0 \;\Rightarrow\; c_2^-(\mathsf{K}_\Sigma\tilde{c}_1^+ + r^+_{-\infty}) = 0$$

where we used $\tilde{\varrho}^+\kappa_2 = -\tilde{\varrho}^-$ and $\tilde{c}_2^+ = (\tilde{c}_2^+)^\dagger$. On account of $\tilde{c}_2^+ + \tilde{c}_2^- = \mathbb{1}$

$$\begin{aligned}\tilde{c}_2^+\mathsf{K}_\Sigma =& (\mathbb{1} - \tilde{c}_2^-)\mathsf{K}_\Sigma(\tilde{c}_1^+ + \tilde{c}_1^-) = \mathsf{K}_\Sigma\tilde{c}_1^+ - \tilde{c}_2^-\mathsf{K}_\Sigma\tilde{c}_1^+ + (\mathbb{1} - \tilde{c}_2^-)\mathsf{K}_\Sigma\tilde{c}_1^- =\\ =& \mathsf{K}_\Sigma\tilde{c}_1^+ + r^+_{-\infty} + c_2^+\mathsf{K}_\Sigma\tilde{c}_1^- = \mathsf{K}_\Sigma\tilde{c}_1^+ + r^+_{-\infty} + r^-_{-\infty} = \mathsf{K}_\Sigma\tilde{c}_1^+ + K^+_{-\infty}\end{aligned}$$

□

# SOURCE OF THE SMOOTHING OBSTRUCTION: SKETCH OF THE PROOF

(viii) Let $f \in \operatorname{Ker} \mathrm{K}_\Sigma^\dagger|_{C_c^\infty}$. With a gauge transformation we can find $h$ and $k$ s.t.

$$\tilde{c}_2^+ f = k + \mathrm{K}_\Sigma \tilde{c}_1^+ h \qquad \text{and} \qquad \begin{cases} k_{s\Sigma} = 0, \\ \tilde{l}_2 k = k \end{cases}$$

Using the almost gauge invariance + other properties of the Calderón op.

$$q_{2,\mathrm{I},\Sigma}(f|\tilde{c}_2^+ f) = q_{2,\mathrm{I},\Sigma}(\tilde{c}_2^+ f|\tilde{c}_2^+ f) = q_{2,\Sigma}(k|k) + q_{1,\Sigma}(\tilde{c}_1^+ K_{-\infty}^{+\dagger} f|h) + q_{1,\Sigma}(h|\tilde{c}_1^+ K_{-\infty}^{+\dagger} f)$$

Now define $\tilde{k} := \tilde{c}_2^+(f - K_\Sigma \tilde{c}_1^+ h)$. Then we can show

$$\tilde{k} = \tilde{\varrho}_2^+ v \qquad \text{for} \qquad v = \tilde{\mathrm{D}}_{2\Omega}^{-1} \tilde{\varrho}_2^* \tilde{\sigma}_2 (f - K_\Sigma h)$$

Since $\tilde{\mathrm{D}}_2 v = 0$ in $\Omega^+$ with $v\restriction_{\partial\Omega^+\setminus\Sigma} = 0$ we obtain by Green's formula that

$$\tilde{q}_2(\tilde{k}|\tilde{k}) = 2\operatorname{Re} Q_{\Omega^+}(v, v) \geqslant 0,$$

where the positivity follows from coercivity of $Q_{\Omega^+}$. But since

$$k = \tilde{k} - \tilde{c}_2^- K_\Sigma \tilde{c}_1^+ h = \tilde{k} - \tilde{c}_2^- K_{-\infty}^+ h.$$

then we can construct a smoothing operator s.t. $\mathrm{q}_{2,I,\Sigma}(\mathfrak{f}|c_2^\pm + \tilde{r}_{2,-\infty}\mathfrak{f}) \geqslant 0$ □

## NO-GO THEOREM FOR CONFORMAL BOUNDARY CONDITIONS

DEFINITION: A boundary conditions for $D_2$ is said to be **gauge invariant** if:
$\forall\ \omega$ s.t. $D_1\omega = 0$ near $\partial\Omega$ and $\omega|_{\partial\Omega}$ satisfies a boundary condition for $D_1$
$\Rightarrow u := K\omega$ satisfies boundary conditions for $D_2$.

THEOREM [M. Capoferri, S.M., G. Schimd]

Let $(\Sigma, \gamma)$ be a complete Riemannian 3-manifold with $\partial\Sigma = \emptyset$
Define $\Omega := [-T, T] \times \Sigma$ with the metric $g := \mathrm{d}s^2 + \gamma$
Suppose that $\mathrm{Ric}(\gamma) = 0$ and there exist non-trivial $L^2$-harmonic 1-forms on $\Sigma$.
If $D_2$ is coupled with a first-order, elliptic and gauge invariant b.c. including

$$\delta u = 0 \qquad \text{and} \qquad u_{\Sigma\Sigma} = \frac{1}{3}\mathrm{tr}_\gamma(u_{\Sigma\Sigma})\gamma \quad \text{on} \quad \partial\Omega = \{\pm T\} \times \Sigma\,,$$

then $0 \in \sigma(D_2)$.

WHAT ARE GOOD BOUNDARY CONDITIONS FOR GRAVITY?

THANKS for your attention!