On the Cauchy problem for symmetric hyperbolic systems on globally hyperbolic manifolds with timelike boundary

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Outline of the Talk

- **Preliminaries**
    - Globally hyperbolic manifolds with timelike boundary
    - Symmetric hyperbolic systems

- **Boundary conditions**
    - Admissible boundary conditions
    - Self-adjoint elliptic boundary conditions

- **Energy estimates**

- **Existence and uniqueness of smooth solutions**

- **Outlook**

Working in progress

N. Drago, N. Große "Classical Dirac operator with selfadjoint elliptic boundary conditions"

N. Ginoux "Admissible boundary conditions for symmetric hyperbolic systems"

Globally hyperbolic manifolds with timelike boundary

- M is connected, time-oriented, oriented smooth manifold with boundary ∂M

- g and $g|_{\partial M}$ are Lorentzian metric $\implies M$ is Lorentzian with timelike ∂M

Few definitions:

- *Temporal function:* $t \in C^\infty(M, \mathbb{R})$ strictly increasing on future directed causal curve and
∇t is timelike everywhere and past-pointing

- *Cauchy hypersurface* Σ: if each inextendible timelike curve $\gamma \cap \Sigma = \{pt\}$

- *Globally hyperbolic:* M *strongly causal* and $\forall p, q \in M, J^+(p) \cap J^-(q)$ compact

> **Bernal and Sánchez (2005) – Aké, Flores and Sánchez (2019):**
>
> M is globally hyperbolic (with timelike boundary)
>
> $\Updownarrow$
>
> Exists a Cauchy temporal function ($t^{-1}(s) := \Sigma_s$ is a Cauchy) and $\nabla t \in T\partial M$
>
> $\Downarrow$
>
> M isometric to $\mathbb{R} \times \Sigma$ with metric $-\beta^2 dt^2 + h_t$, where $\beta \in C^\infty(M, (0, \infty))$

Example: Minkoski spacetime ($\mathbb{R}^4, \eta$), Schwarzchild spacetime ($\mathbb{R}^2 \times \mathbb{S}^2, g_S$)

NOT Example: anti-de Sitter space ($\mathbb{S}^1 \times \mathbb{R}^3, g_{adS}$), Gödel universe ($\mathbb{R}^4, g_G$)

Symmetric hyperbolic systems

- $E \rightarrow M$ be a $\mathbb{K}$-vector bundle with finite rank N and sesquilinear fiber metric $\prec \cdot \mid \cdot \succ$

Definition: a 1st order S is called **symmetric hyperbolic system** if

(S) $\sigma_S(\xi) \colon E_p \rightarrow E_p$ is Hermitian with respect to $\prec \cdot \mid \cdot \succ$, $\forall \xi \in T_p^* M$ and $\forall p \in M$.

(H) $\prec \sigma_S(\tau) \cdot \mid \cdot \succ$ is positive definite on E_p, for any future-directed timelike $\tau \in T_p^* M$

Furthermore S is **of constant characteristic** if $\dim \ker \sigma_S(\mathrm{n}^\flat)$ is constant, where $\mathrm{n} \perp \partial M$.

Example: $E = \mathbb{C}^N \times \mathbb{R}^n \rightarrow (\mathbb{R}^n, \eta)$ with $\prec \mid \succ = \langle \mid \rangle_{\mathbb{C}^N}$

$$S := A_0(p)\partial_t + \sum_{j=i}^{n} A_j(p)\partial_{x_j} + B(p)$$

(S) $A_0 = A_0^\dagger$, $A_j = A_j^\dagger$ (H) $\sigma_S(dt + \sum_j \alpha_j dx_j) = A_0 + \sum_{j=1} \alpha_j A_j > 0$.

Lemma: If $\prec \cdot \mid \cdot \succ$ is indefinite and S is a symmetric hyperbolic system, then:

(I) $\langle \cdot \mid \cdot \rangle := \prec \sigma_S(dt) \cdot \mid \cdot \succ$ is positive Hermitian metric;

(II) $\mathfrak{S} = -\sigma_S(dt)^{-1}S$. is symmetric hyperbolic system

(III) Cauchy problem for $\mathfrak{S}$ is equivalent to the Cauchy problem for S.

Example I: The classical Dirac operator

- $M = (M, g)$ is a globally hyperbolic spin manifold with timelike boundary;
- $\mathbb{S}M$ is a *spinor bundle*: $\mathbb{C}$-vector bundle with N-dim. fibers with indefinite sesquilinear metric

$$\prec \cdot \mid \cdot \succ \colon \mathbb{S}_p M \times \mathbb{S}_p M \to \mathbb{C}$$

and a Clifford multiplication, i.e. fiber-preserving map $\gamma \colon TM \to \mathrm{End}(\mathbb{S}M)$

Dirac operator: $\mathrm{D} := \gamma \circ \nabla^{\mathbb{S}} \colon \Gamma(\mathbb{S}M) \to \Gamma(\mathbb{S}M)$ which in local coordinates reads

$$\mathrm{D} = \sum_{\mu=0}^{n} \varepsilon_\mu \gamma(e_\mu) \nabla^{\mathbb{S}}_{e_\mu}$$

- $(e_\mu)_{\mu=0,\ldots,n}$ is a local orthonormal Lorentzian frame of TM and $\varepsilon_\mu := g(e_\mu, e_\mu)$
- $\gamma(u)\gamma(v) + \gamma(v)\gamma(u) = -2g(u, v)$ for every $u, v \in T_p M$ and $p \in M$.

Remarks:

(i) Topological obstruction to existence of a spinor bundle;

(ii) Existence of spinor bundles on parallelizable manifolds;

(iii) D is nowhere characteristic.

Example II: The geometric wave operator

-M is a globally hyperbolic with timelike boundary and $g = -\beta^2 dt^2 + h_t$;

- V be an Hermitian vector bundle of finite rank;

- P is a normally hyperbolic operator, i.e. $P = \nabla^* \nabla + c$ and principal symbol σ_P defined by

$$\sigma_P(\xi) = -g(\xi, \xi) \cdot \mathrm{Id}_V, \qquad \text{for every } \xi \in T^*M.$$

A norm. hyp. op. P can be reduced to S : $\Gamma(E) \to \Gamma(E)$ defined by

$$S := (A_0 \nabla_{\partial_t} + A_\Sigma \nabla^\Sigma + C)$$

$$\Psi := \begin{pmatrix} \nabla_{\partial_t} u \\ \nabla^\Sigma u \\ u \end{pmatrix} \quad A_0 := \begin{pmatrix} \frac{1}{\beta^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A_\Sigma = \begin{pmatrix} 0 & -\mathrm{tr}_{h_t} & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} \text{suitable} \end{pmatrix}.$$

Remarks:

(i) The Cauchy problem for P can be made equivalent to the Cauchy problem for S;

(ii) S is of constant characteristic,

$$\sigma_S(\mathbf{n}^\flat) = \begin{pmatrix} 0 & -\mathbf{n}^\flat \lrcorner & 0 \\ -\mathbf{n}^\flat \otimes & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

Admissible boundary conditions

Definition: A boundary space B_{adm} for S is called **admissible** if

- The quadratic form $\Psi \mapsto \prec \sigma_S(n^\flat)\Psi \mid \Psi \succ$ is positive semi-definite on B_{adm};

- rank $B_{adm} = \#$ pointwise non-negative eigenvalues of $\sigma_S(n^\flat)$ counting multiplicity.

The adjoint boundary space is defined by $B_{adm}^\dagger := \left(\sigma_{\mathbf{S}}(n^\flat)(B_{adm})\right)^\perp$, i.e.

$$\{\Phi \in \Gamma(E|_{\partial M}) \mid \text{for any } \Psi \in B_{adm} \text{ it holds } \prec \sigma_{\mathbf{S}}(n^\flat)\Psi \mid \Phi \succ = 0\}.$$

Examples for classical Dirac operators:

Lorentzian MIT boundary space is the range of $\pi_{\mathrm{Lor}} := \frac{1}{2}\left(\mathrm{Id} \pm \imath\gamma(\mathbf{n})\right)$

$$\prec \sigma_{\mathbf{D}}(n^\flat)\pi_{\mathrm{MIT}}\psi \mid \pi_{\mathrm{MIT}}\psi \succ = \prec \gamma(\mathbf{n})\pi_{\mathrm{MIT}}\psi \mid \pi_{\mathrm{MIT}}\psi \succ = \imath \prec \pi_{\mathrm{MIT}}\psi \mid \pi_{\mathrm{MIT}}\psi \succ$$

Riemannian MIT boundary space is the range of $\pi_{\mathrm{Riem}} := \frac{1}{2}\left(\mathrm{Id} - \frac{1}{\beta}\gamma(\mathbf{n})\gamma(\partial_t)\right)$

$$\prec \sigma_{\mathbf{D}}(n^\flat)\pi_{\mathrm{Riem}}\psi \mid \pi_{\mathrm{Riem}}\psi \succ = \prec \gamma(\mathbf{n})\pi_{\mathrm{Riem}}\psi \mid \pi_{\mathrm{Riem}}\psi \succ = \frac{1}{\beta} \prec \gamma(\partial_t)\pi_{\mathrm{Riem}}\psi \mid \pi_{\mathrm{Riem}}\psi \succ \geq 0$$

Admissible boundary conditions

Definition: A boundary space B_{adm} for S is called **admissible** if

- The quadratic form $\Psi \mapsto \prec \sigma_S(n^\flat)\Psi \mid \Psi \succ$ is positive semi-definite on B;

- rank $B_{adm} = \#$ pointwise non-negative eigenvalues of $\sigma_S(n^\flat)$ counting multiplicity.

The adjoint boundary space is defined by $B_{adm}^\dagger := \left(\sigma_S(n^\flat)(B_{adm}) \right)^\perp$, i.e.

$$\{\Phi \in \Gamma(E|_{\partial M}) \mid \text{for any } \Psi \in B_{adm} \text{ it holds } \prec \sigma_S(n^\flat)\Psi \mid \Phi \succ = 0\} \,.$$

Examples for geometric wave operator:

Neumann like-boundary condition: $\quad \nabla_n^\Sigma u = 0 \qquad \implies \quad B_{adm}^{N.} = \ker \begin{pmatrix} 0 & n_\lrcorner & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Transparent boundary condition: $\quad \nabla_n^\Sigma u = -b\nabla_{\partial_t} u \quad \implies \quad B^{T.} = \ker \begin{pmatrix} b & n_\lrcorner & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

NOT example for geometric wave operator:

Robin boundary condition: $\quad \nabla_n^\Sigma u = bu \quad (b \neq 0) \quad \Rightarrow \quad B_{adm}^{R.} = \ker \begin{pmatrix} 0 & n_\lrcorner & -b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(the quadratic form is not positive semi-definite)

Self-adjoint elliptic boundary conditions

- M is a globally hyperbolic spin with timelike boundary and $\partial\Sigma$ is compact;

- Dirac operator on spinor bundle $\mathbb{S}M$ reads as $\mathsf{D} = -\gamma(dt)\nabla_{\partial_t} + \mathsf{D}_\Sigma$

- Dirac Hamiltonian $\mathsf{H} := \imath\gamma(dt)^{-1}\mathsf{D}_\Sigma$ $\qquad$ ($\mathsf{D}\Psi = 0 \iff \imath\nabla_{\partial_t}\Psi = \mathsf{H}\Psi$)

- adapted operator $A : \Gamma(\mathbb{S}M|_{\partial M}) \to \Gamma(\mathbb{S}M|_{\partial M})$ formally self-adjoint s. t. $\forall p \in \partial\Sigma_t \ \forall \xi \in T_p^*\partial\Sigma$

$$\sigma_A(\xi) := \sigma_{\mathsf{H}(t)}(\nu_t^\flat)^{-1} \circ \sigma_{\mathsf{H}(t)}(\xi) \qquad \sigma_{\mathsf{H}(t)}(\nu_t^\flat) \circ A = -A \circ \sigma_{\mathsf{H}(t)}(\nu_t^\flat),$$

- Sobolev-like spaces $\qquad \breve{\mathcal{H}}(A(t)) := \mathcal{H}^{\frac{1}{2}}_{(-\infty,a)}(A(t)) \oplus \mathcal{H}^{-\frac{1}{2}}_{[a,\infty)}(A(t))$

$$\mathcal{H}^s_I(A(t)) := \left\{ \sum_j \alpha_j \varphi_j(t) \in L^2(\mathbb{S}M|_{\partial\Sigma_t}) \ \Big| \ \sum_{j|\lambda_j \in I} |\alpha_j|^2 (1+\lambda_j^2(t))^s < +\infty \right\}, \quad s \in \mathbb{R},$$

Definition: A boundary space B_{ell} for D is called **self-adjoint elliptic** if

(B) $\mathsf{B}_{ell}(t) \subset \breve{\mathcal{H}}(A(t))$ is closed;

(E) $\Psi \in \mathcal{H}^k_{loc}(\mathbb{S}M|_{\Sigma_t}) \iff \mathsf{H}_{ell}(t)\Psi \in \mathcal{H}^{k+1}_{loc}(\mathbb{S}M|_{\Sigma_t})$,

(SA) $\mathsf{B}_{ell}(t) = \mathsf{B}_{ell}^\dagger(t) := \{\Phi \in \breve{\mathcal{H}}(A(t)) \,|\, \text{for any } \Psi \in \mathsf{B}_{ell} \text{ it holds } \prec \sigma_{\mathsf{H}(t)}(\mathsf{n}^\flat)\Psi \,|\, \Phi \succ = 0\}$

Examples:

APS boundary conditions $\mathsf{B}_{APS} := \mathcal{H}^{\frac{1}{2}}_{(-\infty,\mathbf{0})}(A(t))$ (self-adjoint $\iff \ker A(t) = \{0\}$).

TAKE HOME MESSAGES

Theorem [Ginoux-M.]

- $M = \mathbb{R} \times \Sigma$ globally hyperbolic with timelike boundary

There exists a **unique strong solution** to the Cauchy problem for a symmetric hyperbolic system

$$\begin{cases} S\Psi = \mathfrak{f} \\ \Psi|_{E|_{\Sigma_0}} = \mathfrak{h} \\ \Psi \in B_{adm} \end{cases}$$

If $(\mathfrak{f}, \mathfrak{h}) \in \Gamma(E) \times \Gamma(E|_{\Sigma_0})$ satisfy compatibility conditions

(i) S is nowhere characteristic $\implies$ the solution is **smooth**;

(ii) S is of constant characteristic $\implies$ the solution is **smooth in tangential direction**.

Theorem [Drago-Große-M.]

- $M = \mathbb{R} \times \Sigma$ globally hyperbolic with timelike boundary with $\partial\Sigma$ compact

There exists a **unique smooth solution** Ψ to the Cauchy problem for the Dirac operator

$$\begin{cases} D\Psi = \mathfrak{f} \\ \Psi|_{\mathbb{S}M|_{\Sigma_0}} = \mathfrak{h} \\ \Psi \in B_{ell} \end{cases}$$

where $(\mathfrak{f}, \mathfrak{h}) \in \check{\Gamma}_c(\mathbb{S}M) \times \check{\Gamma}_c(\mathbb{S}M|_{\Sigma_0}) \subset \Gamma(\mathbb{S}M) \times \Gamma(\mathbb{S}M|_{\Sigma_0})$.

Energy estimates

Theorem (Energy estimates)

- $M = \mathbb{R} \times \Sigma$ globally hyperbolic with timelike boundary (and $\partial\Sigma$ compact for elliptic B.C.)
- S is symmetric hyperbolic with B_{adm} while D is Dirac operator with B_{ell}

Then for each $t_0 \in t(M)$ there exists constants $C > 0$ such that for all $t_1 \geq t_0$ it holds

$$\int_{\Sigma_{t_1}} |\Psi|^2 d\mu_{t_1} \leq Ce^{C(t_1-t_0)} \int_{t_0}^{t_1} \int_{\Sigma_s} |D\Psi|^2 d\mu_s ds + e^{C(t_1-t_0)} \int_{\Sigma_{t_0}} |\Psi|^2 d\mu_{t_0} \qquad \forall\Psi \in B_{ell}$$

$$\int_{\Sigma_{t_1}^p} |\Psi|^2 d\mu_{t_1} \leq Ce^{C(t_1-t_0)} \int_{t_0}^{t_1} \int_{\Sigma_s^p} |S\Psi|^2 d\mu_s ds + e^{C(t_1-t_0)} \int_{\Sigma_{t_0}^p} |\Psi|^2 d\mu_{t_0} \qquad \forall\Psi \in B_{adm}$$

where $t\colon M \to \mathbb{R}$ be a Cauchy temporal function and $\Sigma_s^p := J^-(p) \cap \Sigma_s$

Corollary (uniqueness)

If there exists a solution to the Cauchy problem with admissible/elliptic B.C., then it is **unique**

Proof: If $S(\Psi - \Phi) = 0$, $\Psi, \Phi \in B_{adm/ell}$ *and* $\Psi|_{\Sigma_0} = \Phi|_{\Sigma_0} \xrightarrow{\text{Energy estimates}} \Psi - \Phi = 0$

Corollary (finite speed of propagation with B_{adm})

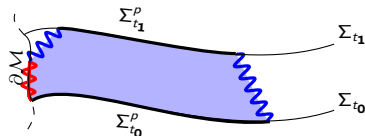$$\text{supp } \Psi \subset \mathcal{V} := J(\text{supp } f) \cup J(\text{supp } h)$$

Proof: If $p \notin \mathcal{V}$ *then* $f|_{M \setminus \mathcal{V}} = 0$ *and* $h|_{M \setminus \mathcal{V}} = 0 \xrightarrow{\text{Energy estimates}} \Psi|_{M \setminus \mathcal{V}} = 0$

Energy estimates (admissible boundary conditions)

<u>Sketch of the proof:</u>

- n-differential form:

$$\omega := \sum_{j=0}^{n} \Re e \left(\prec \sigma_{\mathbf{S}}(b_j^{\flat}) \Psi \mid \Psi \succ \right) b_j \lrcorner \mathrm{vol}_M$$

- Stokes' theorem for manifold with Lipschitz boundary yields

$$\int_K d\omega = \int_{\partial K} \omega = \int_{\Sigma_{t_1}^p} \omega - \int_{\Sigma_{t_0}^p} \omega + \int_{\text{red}} \omega + \int_{\text{blue}} \omega$$

- Hyperbolicity of S $\implies \int_{\text{blue}} \omega \geq 0$ while $\Psi \in B_{adm} \implies \int_{\text{red}} \omega \geq 0$

$$\int_{\Sigma_{t_1}^p} |\Psi|^2 d\mu_1 - \int_{\Sigma_{t_0}^p} |\Psi|^2 d\mu_0 \leq \int_K d\omega \leq C \int_{t_0}^{t_1} \int_{\Sigma_s^p} (|\Psi|^2 + |S\Psi|^2) d\mu_s ds$$

- By setting $\quad h(s) := \int_{\Sigma_s^p} |\Psi|^2 d\mu_s, \quad \alpha(t_1) := C \int_{t_0}^{t_1} \int_{\Sigma_s^p} (|\Psi|^2 + |S\Psi|^2) d\mu_s ds + \int_{\Sigma_0^p} |\Psi|^2 d\mu_0$

and using Grönwall, we obtain: $\quad h(t_1) \leq \alpha(t_1) + C \int_{t_0}^{t_1} h(s) ds \leq \alpha(t_1) e^{C(t_1 - t_0)}$

$\square$

Weak and Strong Solutions in a time strip $M_T := t^{-1}(t_1, t_0)$

Definition: We call $\Psi \in \mathscr{H} := \overline{(\Gamma_c(E|_{M_T}), (\cdot \mid \cdot)_{M_T})}^{(\cdot \mid \cdot)_{M_T}}$

(W) **Weak Solution** if it holds $(\Phi \mid \mathfrak{f})_{M_T} = (S^\dagger \Phi \mid \Psi)_{M_T}$

for any $\Phi \in \Gamma_c(E|_{M_T})$ such that $\Phi \in B^\dagger_{adm/ell}$ and $\Phi|_{\Sigma_{t_1}} = 0 = \Phi|_{\Sigma_{t_0}}$

(S) **Strong Solution** if $\exists \{\Psi_k\}_k$, $\Psi_k \in \Gamma(E|_{M_T})$ s.t. $\Psi_k \in B_{adm/ell}$ on ∂M and

$$\|\Psi_k - \Psi\|_{L^2(M_T)} \xrightarrow{k \to \infty} 0 \quad \text{and} \quad \|S\Psi_k - \mathfrak{f}\|_{L^2(M_T)} \xrightarrow{k \to \infty} 0$$

Theorem (weak=strong with admissible boundary conditions B_{adm})

- $M = \mathbb{R} \times \Sigma$ globally hyperbolic with timelike boundary

- S is symmetric hyperbolic with admissible boundary conditions B_{adm}

Any weak solution is a strong solution. Moreover, if $(\mathfrak{f}, \mathfrak{h})$ satisfy compatibility conditions

(i) S is nowhere characteristic $\implies$ the solution is **smooth**;

(ii) S is of constant characteristic $\implies$ the solution is **smooth in tangential direction**;

<u>Comments on the Proof</u>

- Admissible boundary conditions are local, so we can localise

- In Fermi coordinates, we can use the local theory [Phillips-Lax,Rauch,Massey-Rauch].

Existence of a Weak Solution

> **Theorem** (existence weak solutions)
>
> - $M = \mathbb{R} \times \Sigma$ globally hyperbolic with timelike boundary (and $\partial\Sigma$ compact for elliptic B.C.)
> - S is symmetric hyperbolic with B_{adm} or is Dirac operator with B_{ell}
>
> There exists a weak solution to the Cauchy problem.

Sketch of the proof:

- Energy Estimates: $\|\Phi\|_{L^2(M_T)} \leq c\|S^\dagger\Phi\|_{L^2(M_T)}$

- The kernel of the operator $S^\dagger$ acting on dom $S^\dagger$ is trivial

$$\text{dom } S^\dagger := \{\Phi \in \Gamma_c(E_{M_T}) \mid \Phi|_{\Sigma_{t_1}} = 0, \Phi|_{\Sigma_{t_0}} = 0, \Phi \in B_{adm/ell}\}$$

- $\ell \colon S^\dagger(\text{dom } S^\dagger) \to \mathbb{C}$ given by $\ell(\Theta) = (\Phi \mid \mathfrak{f})_{M_T}$ where Φ satisfies $S^\dagger\Phi = \Theta$

- Energy Estimates $\Rightarrow \ell$ is bounded:

$$\ell(\Theta) = (\Phi \mid \mathfrak{f})_{M_T} \leq \|\mathfrak{f}\|_{L^2(M_T)} \|\Phi\|_{L^2(M_T)} \qquad \textbf{(Cauchy-Schwarz inequality)}$$

$$\leq \lambda^{-1}\|\mathfrak{f}\|_{L^2(M_T)}\|S^\dagger\Phi\|_{L^2(M_T)} = \lambda^{-1}\|\mathfrak{f}\|_{L^2(M_T)}\|\Theta\|_{L^2(M_T)},$$

-Hence $(\Phi \mid \mathfrak{f})_{M_T} = \ell(\Theta) \overset{Riesz}{\underset{Thm.}{=\!=}} (\Theta \mid \Psi)_{M_T} = (S^\dagger\Phi \mid \Psi)_{M_T}$ for all $\Phi \in \text{dom} S^\dagger$

$\square$

Existence of smooth solutions with elliptic boundary conditions

Theorem (existence smooth solutions with elliptic boundary conditions B_{ell})

- $M = \mathbb{R} \times \Sigma$ globally hyperbolic with timelike boundary and $\partial\Sigma$ compact
- D is a Dirac operator with self-adjoint elliptic boundary condition B_{ell}

There exists a **unique smooth solutions** to the Cauchy problem.

<u>Sketch of the proof</u>

- By finite speed of propagation in the bulk, Σ can be chosen compact

- Sobolev-like spaces $\mathcal{H}_{\mathrm{bc}}^\infty(E|_{\Sigma_t}) := \bigcap_k \left(\mathcal{H}_{\mathrm{bc}}^k(E|_{\Sigma_t}) := \mathrm{dom}\langle \mathsf{H}_{ell}(t)^2 + 1 \rangle^{\frac{k}{2}} \right) \subset \Gamma(E)$

- Mollifier $J_\varepsilon(t) := \exp[-\varepsilon\langle \mathsf{H}_{ell}(t)^2 + 1 \rangle] \colon \mathcal{H}_{\mathrm{bc}}^\infty(E|_{\Sigma_t}) \to \mathcal{H}_{\mathrm{bc}}^\infty(E|_{\Sigma_t})$

- Existence of unique solution of regularized equation $\imath \nabla_{\partial_t} \Psi^{(\varepsilon)} = J_\varepsilon(t) \mathsf{H}_{ell}(t) J_\varepsilon(t) \Psi^{(\varepsilon)} + f$,

- Estimates ($\Rightarrow$ equicontinuity) + Ascoli-Arzela ($\Rightarrow$ relatively compactness) + diag. subseq. arg.

  $\{\Psi^{(\varepsilon)}\}_{\varepsilon > 0} \supset \{\Psi^{(\varepsilon_j)}\}_{j \in \mathbb{N}} \xrightarrow{j \to +\infty} \Psi \in C^1(\mathbb{R}, \mathcal{H}_{\mathrm{bc}}^\infty(E|_{\Sigma_t}))$ and satisfies $\imath \nabla_{\partial_t} \Psi = \mathsf{H}_{ell}\Psi + f$

 - By showing that also $\nabla_{\partial_t} \Psi \in C^0(\mathbb{R}, \mathcal{H}_{\mathrm{bc}}^\infty(E|_{\Sigma_t}))$ and iterating, we can conclude. $\qquad\square$

Outlook

WHAT WE HAVE SEEN AND WHAT COMES NEXT?

well-posedness of the Cauchy problem for

    ✓ Dirac operator with self-adjoint elliptic b. c. (Nicolò Drago & Nadine Große)

        e.g. classical Dirac operator with APS boundary condition

    ✓ Symmetric hyperbolic systems with admissible boundary condition (Nicolas Ginoux)

        e.g. Wave equation with Neumann and transparent boundary condition

        e.g. Classical Dirac operator with MIT boundary conditions

What comes next?

    ? De Rham-d'Alembert operator (with Nicolas Ginoux)

    ? Dirac type-operator with general elliptic boundary condition

THANKS for your attention!