A new construction of algebraic states for CAR algebras

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Dirac Fields

Kinematics and Dynamics

Let $SM \equiv SM[\mathbb{C}^4, \pi, M]$ and $S^*M \equiv S^*M[(\mathbb{C}^4)^*, \pi, M]$ be the spinor and cospinor bundles over a spacetime (M, g). We introduce bundle isomorphisms on the **kinematic configurations** :

$$A\,:\;\; \Gamma^\infty(SM) \to \Gamma^\infty(S^*M), \qquad \psi \;\mapsto\; A\psi := (\overline{\psi})^T\gamma_0$$

and a Lorentz invariant pairing $\langle\,|\,\rangle\colon \Gamma^\infty_{\mathrm{c}}(S^*M) \times \Gamma^\infty(SM) \to \mathbb{R}$

$$\langle f \mid \psi \rangle \doteq \int_M f(\psi)\, d\mu_g.$$

The dynamics is ruled by the **Dirac operator** on SM and its dual on S^*M:

$$\mathcal{D}\psi_m \doteq (i\gamma^\mu\nabla_\mu - m)\psi_m = 0, \qquad \mathcal{D}^*\phi_m = (-i\gamma^\mu\nabla_\mu - m)\phi_m = 0.$$

We call **causal propagators** $E^{(*)} : \Gamma^\infty_{\mathrm{c}}(S^{(*)}M) \to \Gamma^\infty_{\mathrm{sc}}(S^{(*)}M)$

$$\mathcal{D}^{(*)} \circ E^{(*)} = 0 = E^{(*)} \circ \mathcal{D}^{(*)}|_{\Gamma^\infty_{\mathrm{c}}(S^{(*)}M)}$$

$$supp(E^{(*)}(f)) \subseteq J^\pm(supp(f)), \qquad \forall f \in \Gamma^\infty_{\mathrm{c}}(S^{(*)}M)$$

that characterise the **dynamical configurations**

$$\mathcal{S}ol(\mathcal{D}) \simeq \frac{\Gamma^\infty_{\mathrm{c}}(M, SM)}{\mathcal{D}\Gamma^\infty_{\mathrm{c}}(M, SM)} \qquad \mathcal{S}ol(\mathcal{D}^*) \simeq \frac{\Gamma^\infty_{\mathrm{c}}(M, S^*M)}{\mathcal{D}^*\Gamma^\infty_{\mathrm{c}}(M, S^*M)}.$$

Introducing the **scalar products** $(\,|\,)^s_m$ on $\mathcal{S}ol(\mathcal{D})$ and $(\,|\,)^c_m$ on $\mathcal{S}ol(\mathcal{D}^*)$

$$\left(\psi_m \mid \widetilde{\psi}_m\right)^s_m \doteq \int_\Sigma A\psi_m(\not{\nu}\,\widetilde{\psi}_m)\, d\Sigma, \qquad \left(\phi_m \mid \widetilde{\phi}_m\right)^c_m \doteq \int_\Sigma \phi_m(\not{\nu}\, A^{-1}\widetilde{\phi}_m)\, d\Sigma,$$

we obtain the **Hilbert spaces**

$$\mathcal{H}^s_m := \overline{\left(\mathcal{S}ol(\mathcal{D}), (\,|\,)^s_m\right)} \qquad \mathcal{H}^c_m := \overline{\left(\mathcal{S}ol(\mathcal{D}^*), (\,|\,)^c_m\right)}.$$

Quantum Field Theory

- **Field algebra** $\mathcal{F}$: the unital $*$-algebra generated by the abstract elements $1_\mathcal{F}$, $\Phi(\psi_m)$ and $\Psi(\phi_m)$, together with the following relations:

(i) $\Phi(\alpha\psi_m + \beta\widetilde{\psi}_m) = \alpha\Phi(\psi_m) + \beta\Phi(\widetilde{\psi}_m)$,

(ii) $\Phi(\psi_m)^* = \Psi(A\psi_m)$,

(iii) $\left\{\Phi(\psi_m), \Phi(\widetilde{\psi}_m)\right\} = 0 = \left\{\Psi(\phi_m), \Psi(\widetilde{\phi}_m)\right\}$ and $\left\{\Psi(\phi_m), \Phi(\psi_m)\right\} = \left(A^{-1}\phi_m \mid \psi_m\right)^s_m \cdot 1_\mathcal{F}$.

- **Algebraic state** ω: a complex valued, linear functional on $\mathcal{F}$ such that

$$\omega(1_\mathcal{F}) = 1, \qquad \omega(h^*h) \geq 0 \quad \forall h \in \mathcal{F}.$$

Theorem: Every projector operator P on the Hilbert space $\mathcal{H}^s_m$ defines a quasi-free algebraic state on the field algebra $\mathcal{F}$:

$$\omega_2\big(\Psi(\phi_m)\Phi(\psi_m)\big) =: \left(A^{-1}\phi_m \,|\, P\psi_m\right)_s.$$

Microlocal Analysis and Hadamard States

We call **space of symbols**

$$S^\lambda := \Big\{ q \in C^\infty(U \subset \mathbb{R}^n \times \mathbb{R}^n) \,:\, |\, D^\alpha_\xi\, q(x,\xi)\,| \leq C_{\alpha,K}\,(1 + |\,\xi\,|\,)^{\lambda - |\alpha|}$$

$$\forall \alpha \in \mathbb{N}^n,\ \forall K \subset \mathbb{R}^n \text{ compact and } x \in K,\ \xi \in \mathbb{R}^n,\ C_{\alpha,K} \in \mathbb{R} \Big\}.$$

An operator $Q : C^\infty_c(U) \to C^\infty(U)$ is called **pseudo-differential** of degree λ if can be written as

$$Qu(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(x,\xi)\hat{u}(\xi)\, d\xi, \qquad u \in C^\infty_c(U),$$

where $q(x,\xi) \in S^\lambda$. We define the **wavefront set** for $u \in D'(U)$:

$$\mathrm{WF}(u) := \bigcap_{Qu \in C^\infty(U)} \Big\{ (x,\xi) \in U \times \mathbb{R}^n \setminus 0 : q(x,\xi) = 0 \Big\}.$$

If $\Xi : V \to U$ is a diffeomorphism, the pull-back distribution $\Xi^* u \in D'(V)$ fulfils:

$$\mathrm{WF}(\Xi^* u) = \Xi^*\mathrm{WF}(u) := \Big\{ (\Xi^* x, \Xi^* \xi) : (x,\xi) \in \mathrm{WF}(u) \Big\}.$$

We can extend therefore the definition of WF to distributions on the spinor bundle SM to be

$$\mathrm{WF}(u) := \bigcup_i \mathrm{WF}(u_i).$$

A (quasi-free) state ω satisfies the **Hadamard condition** if and only if

$$WF(\omega_2) = \Big\{ (x, y, \xi_x, \xi_y) \in T^*M^{\otimes 2} \setminus 0 \;\Big|\; (x,\xi_x) \sim (y, -\xi_y), \quad \xi_x \rhd 0 \Big\},$$

where $(x,\xi_x) \sim (y, -\xi_y)$ implies that x and y are connected by a null geodesic and $-\xi_y$ is the parallel transport of the co-parallel co-vector ξ_x; whereas $\xi_x \rhd 0$ means that ξ_x is future-pointing.

The Fermionic Projector

We construct **families of solutions** $\Psi := (\psi_m)_{m \in I \subset (0,\infty)}$ of the Dirac equation for a variable mass parameter and we get the Hilbert space:

$$\mathcal{H} := \overline{\left(\Gamma^\infty_{\mathrm{sc,c}}(SM \to M \times I) \;,\; (\,|\,) := \int_I (\,|\,)_m\, dm \right)}.$$

We call **smearing operator** :

$$\mathfrak{p}\;:\; \mathcal{H} \to \Gamma^\infty_{\mathrm{sc}}(SM), \qquad \Psi \;\mapsto\; \mathfrak{p}\Psi := \int_I \psi_m\, dm$$

and **strong mass oscillation property**:

$$\left| \left\langle \mathfrak{p}\Psi \mid \mathfrak{p}\widetilde{\Psi} \right\rangle \right| \leq c \int_I ||\psi_m||_m\, ||\widetilde{\psi}_m||_m\, dm.$$

By the Riesz representation theorem, we get the **fermionic signature operator** S_m:

$$\left\langle \mathfrak{p}\Psi \mid \mathfrak{p}\widetilde{\Psi} \right\rangle = \left(\Psi \mid \mathsf{S}\,\widetilde{\Psi} \right) := \int_I \left(\psi_m \mid \mathsf{S}_m\,\widetilde{\psi}_m \right)_m dm.$$

Using the spectral calculus we realise the **fermionic projector** P as:

$$P := \chi(\mathsf{S}_m) \circ E : \Gamma^\infty_{\mathrm{c}}(SM) \to \mathcal{H}^s_m.$$

Fermionic Projector in Minkowski spacetime

We restricted our attention to subsets of Minkowski spacetime and to the Dirac equation

$$(i\not{\partial} + \mathcal{B} - m)\psi_m = 0.$$

Theorem:

$$|\mathcal{B}(t)|_{C^2} \leq \frac{c}{1 + |t|^{2+\epsilon}}, \qquad \Longrightarrow \qquad \text{strong mass oscillation property.}$$

We decompose the fermionic signature operator S_m as

$$\mathsf{S}_m = \mathsf{S}^D + \Delta\mathsf{S}\,, \qquad \text{where} \qquad \mathsf{S}^D := \mathsf{S}^+_+ + \mathsf{S}^-_- \quad \text{and} \quad \Delta\mathsf{S} := \mathsf{S}^+_- + \mathsf{S}^-_+\,.$$

Theorem:

$$\int_{-\infty}^{\infty} |\mathcal{B}(t)|_{C^0}\, dt < \sqrt{2} - 1 \quad \text{and} \quad \int_{-\infty}^{\infty} |\partial^p_t \mathcal{B}(t)|_{C^0}\, dt < \infty \quad \forall p \in \mathbb{N}$$

$$\Downarrow$$

$$\chi(\mathsf{S}_m) = \chi(H) + \frac{1}{2\pi i} \oint_{\partial B_{\frac{1}{2}}(\pm 1)} (\mathsf{S}_m - \lambda)^{-1}\, \Delta\mathsf{S}\, (\mathsf{S}^D - \lambda)^{-1}\, d\lambda.$$

Theorem:

The fermionic projector P satisfies the Hadamard condition.

References

[1] M. Benini and C. Dappiaggi, *Models of free quantum field theories on curved backgrounds*, Advances in Algebraic Quantum Field Theory, Springer, 2015, pp. 73–122.

[2] F. Finster, S. Murro, and C. Röken, *The fermionic projector in a time-dependent external potential: Mass oscillation property and Hadamard states*, arXiv preprint arXiv:1501.05522 (2015).

[3] F. Finster and M. Reintjes, *A non-perturbative construction of the fermionic projector on globally hyperbolic manifolds II -space-times of infinite lifetime*, arXiv preprint arXiv:1312.7209 (2013).

[4] L. Hörmander, *Linear differential operators*, Proc. Nice Congress, vol. 1, 1970, pp. 121–133.

[5] I. Khavkine and V. Moretti, *Algebraic QFT in Curved Spacetime and quasifree Hadamard states: an introduction*, Advances in Algebraic Quantum Field Theory, Springer, 2015, pp. 187–246.

[6] M.J. Radzikowski, *Micro-local approach to the Hadamard condition in quantum field theory on curved space-time*, Communications in Mathematical Physics **179** (1996), no. 3, 529–553.