QUANTUM STATES ON THE ALGEBRA OF DIRAC FIELDS:

A FUNCTIONAL ANALYTIC APPROACH

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Dedicated to my parents in deep gratitude
and to the memory of Rudolf Haag.

Abstract

The aim of this thesis is to use functional analytic techniques to construct quasifree states on the algebras of observables for massive Dirac fields. We begin by considering the Rindler spacetime. In the two-dimensional setting, the resulting quasifree states coincide with the Fulling-Rindler vacuum and the Unruh state. On the other hand, in the four-dimensional case new quantum states arise. In more general spacetimes, we focus our analysis on families of solutions for the Dirac equation with a varying mass parameter. By introducing a sequence of Møller-like operators, we are able to construct a class of Fermionic signature operators, each of those generates a quantum state. As a final result, we realize an isomorphism between the algebra of massless Dirac fields and the massive one. Along this isomorphism we can pull back quasifree states from the former to the latter algebra.

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Chapter 1

Introduction

Quantum field theory on curved spacetimes is a well-established and very promising research field in mathematical physics. Algebraic methods proved to be very successful in this context. The original idea can be tracked in [HK64]. Haag and Kastler realized that a quantum theory could be also understood as an assignment of an algebra of observables on Minkowski spacetime. This algebra should naturally encode both geometric and dynamical features of quantum theory, such as locality, causality, and covariance with respect to the isometry group of Minkowski spacetime. In [Di80], Dimock extended this approach to a more generic class of spacetimes.
To recover the natural probabilistic interpretation, one needs to introduce the notion of algebraic state, which is a positive, linear and normalized functional on the algebra of observables. The value taken by the element of the algebra is interpreted as the mean value of the associated observable. However, not every state can be considered as being of physically relevant. It is widely accepted that a criterion to single out the physical ones is to require the so-called Hadamard condition [GK89, Wa94, FV13]. There are several reasons for this choice: For example, it implies the finiteness of the quantum fluctuations of the expectation value of every observable and it allows to construct Wick polynomials following a covariant scheme, see [HW02] or [KM15] for recent reviews. Thanks to the seminal work of Radzikowski [Ra96a, Ra96b], the Hadamard condition has been translated into the language of microlocal analysis, as a constraint on the wavefront set of the bidistribution associated to the two-point function of the state.

A recent breakthrough in the context of quantum field theory on curved backgrounds is the axiomatic formulation proposed by Brunetti, Fredenhagen, and Verch. In [BFV03] they provided a set of axioms to formalize the concept of quantum field theory over all globally hyperbolic spacetimes at the same time. As a special case, this principle allows to recover the natural generalization to a curved spacetime of the Haag-Kastler axioms. Notably, general local covariance paved the way for the perturbative treatment of interacting fields in the framework of AQFT [HW02, BFK96] eventually leading to new insights about the renormalization of quantum field theories on curved spacetimes [BDF09]. For these reasons, much effort has been spent in the last decade to implement the axioms of general local covariance in concrete models of physical interest, from free field theories to interacting gauge field theories, treated perturbatively using the Batalin-Vilkovisky formalism

[FrRe13]. Recently in [BSS16, BDS14b, BDS14c, FV12a, DMS17] it was shown the failure of the locality axiom of general local covariance for Abelian gauge theories and the impossibility of selecting a single 'natural' state in each spacetime. Between the assumptions of the nonexistance of a natural state, we find that such a state should be determined locally by the geometry. This leaves open the possibility that there might be interesting states determined nonlocally by the geometry.

Results achieved

In this thesis, we investigate a new functional analytic constructions of quasifree states for a massive Dirac field *determined nonlocally by the geometry*. These methods exploit the results in [Ar71], in particular, the one proving that the construction of a projection operator in a Hilbert space is equivalent to the assignment of a pure, quasifree state on a CAR algebra. We begin by considering the two-dimensional Rindler spacetimes, where the solutions of the massive Dirac equation have a good decay in time. For this reason, we can introduce the so-called spacetime inner product, which is defined as the integration over the whole spacetime of pairings of solutions. By fixing one of the entries of this pairing, we obtain a linear map. After showing the boundedness of this map, we apply the Riesz representation theorem to obtain a symmetric, densely defined, known as the fermionic signature operator. This operator turns out to be unbounded: Since all the symmetric, densely defined operator are essentially self-adjoint, we look for an extension. This is achieved once that we work with a plane wave ansatz in a suitable parametrization in momentum space. In fact, this fermionic signature operator becomes a multiplication operator, making it possible to construct a *unique* self-adjoint extension with standard functional analytic methods. Thanks to this ansatz, it is also shown that the fermionic signature operator is a multiple of the Dirac Hamiltonian in Rindler coordinates. This means that the construction of the fermionic signature operator "detects" the Killing symmetry of our spacetime as described by translations in Rindler time. Applying the spectral calculus to the fermionic signature operator, we can construct a spectral projector which gives rise to the Fulling-Rindler vacuum [Fu73]. As a bonus to the unboundedness of the fermionic signature operator, we can also create general thermal states like the Unruh state [Un76]. Surprisingly, after extending the above analysis to four-dimensional Rindler spacetime, the fermionic signature operator is no longer the Dirac Hamiltonian. The states associated with this new operator are indeed different from the Fulling-Rindler vacuum and general thermal states. The physical properties of these new states are still under investigation as well as the Hadamard condition.

As shown in [FiRe16], to cover a more general class of globally hyperbolic spacetimes, we have to extend our analysis to families of solutions of massive Dirac equations. In a few words, it works as follows: Consider both the Dirac equation with the mass m varying parametrically and its smooth solutions, which are spacelike compact as well compact in m. Such a space can be completed to Hilbert space with respect to the scalar product induced by integrating over both the manifold and the mass. Assuming the so-called strong mass oscillation property, which is a constraint on the decay rate of the solutions of the massive Dirac equation at infinity upon integration over the mass, and pairing families of solution integrated over the mass identifies a continuous sesquilinear form on the space of solutions previously introduced. By applying the Riesz theorem, this is tantamount to the assignment of a family of bounded symmetric operators, each acting on the subspace of solutions with a specific, fixed value of the mass. Using the spectral calculus, we can construct a projector operator for fixed values of the mass, dubbed *fermionic projector*. The net advantage of this construction is that it does not depend on any structural property of the underlying background, such as the existence of specific Killing fields. In [FMR16a, FiRe17] this method was applied successfully for a Dirac field on Minkowski spacetime, in the presence of an external time-dependent potential, subject to suitable technical constraints. It is noteworthy that the state obtained satisfies the Hadamard condition. Considerable different are instead the states recently proposed by Afshordi, Aslanbeigi and Sorkin for the real scalar field [AAS12], the so-called 'SJ-states.' In fact, it was shown by explicit computation in [FV12b, FV13] that the SJ-states have some unphysical aspects, e.g., they fail to satisfy the Hadamard condition.

Despite the successes in the 'fermionic projector' program, an undeniable limitation of this method is the intrinsic difficulty in proving that the strong mass oscillation property holds true. It has to be checked case by case and in general, it does not hold true on every spacetime, e.g., Rindler spacetime due to the presence of a horizon. In order to weaken this strict requirement, we investigate an alternative procedure. Our main idea consists of constructing unitary operators that intertwine the dynamics of two Green hyperbolic operators differing only by a mass term, extending thus the work of [DHP17, DD16]. The application of intertwining operators and the integration over the mass on the solution space of the Dirac equation defines a new sesquilinear form, which is continuous in either one or both entries, whenever two modified version of the mass oscillation properties are satisfied. Once more using the Riesz representation theorem, such sesquilinear form yields a symmetric, linear operator on the Hilbert space of the spacelike compact, smooth solutions to the Dirac equation. In addition, still using the results of Araki [Ar71], we construct a pure and quasifree state on the CAR $*$-algebra once again realizing a spectral projector. To prove the robustness of our novel method, we investigate in detail a concrete example in which the strong mass oscillation property does not hold true, but the modified weak one does: a massive Dirac field on Rindler spacetime.

As we shall see in more details later, all these functional analytic methods allow to construct only states for the massive Dirac fields. Even if, massless Dirac particles seem excluded from the Standard Model of elementary particle by experiments, a mathematical genuine question arise: *Given a Hadamard state for the massive Dirac fields, can we build a possible counterpart for the massless case such that this property still holds?* As the last result of this thesis, we address this question. We will show that using the extended Møller-Dappiaggi operator, one can also introduce a deformation argument in mass parameter space. In a few heuristic words, this argument guarantees that, if we can construct a Hadamard state for a free field theory with a fixed value of the mass, then one can induce a counterpart state for the massless case and such state fulfills the Hadamard condition.

Outline

In the following, we summarize the topics investigated in this thesis. In Chapter 2, we set the basis for the subsequent developments. In particular, Section 2.1 introduces globally hyperbolic spacetimes, which provide the background where the field dynamics takes place. We proceed with Section 2.2

where our notation spin bundle is established, spinor and cospinor fields are briefly recalled, together with the Dirac operator and its dual. Section 2.3 is devoted to studying the well-posedness of the Cauchy problem for the Dirac equation. In particular, we characterize the space of solutions using the so-called causal propagator. Section 2.4 deals with the quantization of the Dirac fields adopting an algebraic approach. The algebra of observables for Dirac fields is introduced, and successively the quasifree states over this algebra are characterized. To conclude, we recall the notion of the wavefront set and the Hadamard condition in Section 2.5.

Fock states are constructed in Rindler spacetime in Chapter 3. As a starting point, in Section 3.1 we realize an embedding between the space of solutions in the two-dimensional Rindler spacetime and the one in Minkowski spacetime. In Section 3.2, we construct the relative fermionic signature operator. Working in the momentum space this operator is nothing but a multiplicative operator. With the usual technique of functional analysis, we construct a unique self-adjoint extension. In Section 3.3, we show that the fermionic signature operator is nothing but a multiple of the Dirac Hamiltonian in Rindler spacetime. Taking advantage of this result, we proceed to discuss the associate quasifree state in Section 3.4. To conclude, in Section 3.5 we extend our analysis to the four-dimensional Rindler spacetime.

The extension of this technique to a more general setting is the topic of Chapter 4. In Section 4.1, we extend our analysis to families of solutions for Dirac equations for a varying mass parameter. Furthermore, we state two necessary and sufficient conditions, the so-called mass oscillation properties, under which we can construct a new quasifree state. Before concluding this section, we provide we test our construction in Minkowski and in Rindler spacetime: In the first case, we obtain the vacuum state, while in the latter space we incur in an obstruction. Modifying this technique to remove the obstruction is the goal of Section 4.2.

Since our results apply only to the massive Dirac fields, in Chapter 5, we investigate a method to deform a massive Hadamard state into a massless one. More precisely, in Section 5.1 we realize an isomorphism between the space of massive and massless classical observable for Dirac fields and then we pull back respect this map the massive state in Section 5.2.

CHAPTER 2

CLASSICAL AND QUANTUM DIRAC FIELDS ON LORENTZIAN MANIFOLDS

Since the topic of the current thesis is the construction of quantum states for quantum Dirac fields on Lorentzian manifolds, we begin by introducing and by explaining the necessary structures needed to define in a mathematically rigorous fashion what a quantum Dirac fields on Lorentzian manifolds is. After having set up the geometry of the spacetimes and discussed the well-posedness of the Cauchy problem for the Dirac equation, we proceed to quantizing using the so-called algebraic approach to quantum field theory, which is based on two steps: The first one consists of the assignment to a physical system of a $*$-algebra of observables which encodes structural properties such as causality, dynamics, and the canonical anti-commutation relations. The second step calls for the identification of a quantum state, which is a positive, linear and normalized functional on the algebra of observables. Since not all the quantum states are physically sensible, we will introduce the concept of the wavefront set to formulate the so-called Hadamard condition. With this in mind, let us proceed to present the class of spacetimes we are interested in.

2.1 Globally Hyperbolic Spacetimes

Definition 2.1.1. *A* ***spacetime*** *M is a quadruple $(M,g,\mathfrak{o},\mathfrak{t})$, where:*

- *M is a $n+1$-dimensional Hausdorff, second countable, connected, orientable, time-orientable, smooth manifold, endowed with a smooth Lorentzian metric g of signature $(+,-,\dots,-)$;*
- *$\mathfrak{o}$ is a choice of orientation and $\mathfrak{t}$ is a choice of time-orientation on M.*

The Lorentzian metric g plays a fundamental role in the definition of causal structure. Indeed, one can label a tangent vector $v \in T_xM$ according to the value of $g(v,v)$.

Definition 2.1.2. *We say that a tangent vector $v \in T_xM$ is* ***timelike*** *if $g(v,v) > 0$,* ***lightlike*** *if $g(v,v) = 0$,* ***spacelike*** *if $g(v,v) < 0$, and* ***causal*** *if it is either timelike or lightlike.*

Extending this idea, we call a vector field $v : M \to TM$ *spacelike*, *timelike*, *lightlike*, or *causal* if it possesses this property at each point. Finally, we call a curve $\gamma : [0,1] \to M$, spacelike, timelike, lightlike, or causal if its tangent vector field fulfils this property. A vector field is called *complete* if each of its flow curves exists for all times. As (M,g) is time orientable by assumption, we can say that a causal curve is *future-directed* if $g(\mathfrak{t},\dot{\gamma}) > 0$ where $\dot{\gamma}$ is the tangent vector to the curve and $\mathfrak{t}$ is the time orientation. Conversely if $g(\mathfrak{t},\dot{\gamma}) < 0$, we denote that the curve is past-directed. Hence, given a future directed curve γ parametrised by s. We call y a *future endpoint* of γ if, for every neighbourhood Ω of y, there exists a $\widetilde{s}$ such that $\gamma(s) \in \Omega$ for all $s > \widetilde{s}$. With this in mind, we say that a future directed causal curve is *future inextendible* if, for all possible parametrisations, it has no future endpoint and we define past inextendible, past directed causal curves similarly.

Next, we extend the definition of lightcones to curved spacetimes.

Definition 2.1.3. *We define on a spacetime M the* ***causal future/past*** *of a point x as*

$$J^{\pm}_M(x) := \{y \in M \mid \exists \gamma : I \to M \text{ with } \gamma(0) = x \text{ and } \gamma(1) = y \text{ future-/past-directed and causal curve}\}.$$

Moreover, we define $J_M(\Omega) = J^+_M(\Omega) \cup J^-_M(\Omega)$ and the ***chronological future/past*** *$I^{\pm}_M(x)$ in an analogous way. For a general subset $\Omega \subset M$ we define*

$$I^{\pm}_M(\Omega) = \bigcup_{x\in\Omega} I^{\pm}_M(x), \qquad J^{\pm}_M(\Omega) = \bigcup_{x\in\Omega} J^{\pm}_M(x).$$

The identification of a causal structure suggests us that not all spacetimes should be thought as admissible. Indeed, we could incur in pathological situations such as closed timelike curves. There are plenty of examples available in the literature ranging from the so-called Gödel Universe - see for example [HW97] - to the Anti-de Sitter spacetime - see for example [Mo06]. Therefore it is useful to restrict our attention to a class of spacetimes which avoids such inconveniences while still encompassing interesting curved backgrounds. In order to introduce this class, we need additional structures.

Definition 2.1.4. *Let M be a given spacetime.*

- *A subset $\Sigma \subset M$ is called* ***achronal*** *if $I^+_M(\Sigma) \cap \Sigma = \emptyset$, i.e. every timelike curve in M intersects Σ at most once.*
- *Given a closed achronal set, we call* ***future/past domain of dependence*** *$D^{\pm}_M(\Sigma)$, the collection of all points $y \in M$ such that every past/future inextensible causal curve passing through y intersects Σ.*
- *We say that $\Sigma \subset M$ is a* ***Cauchy surface*** *if it is a closed achronal subset of M such that $D^+_M(\Sigma) \bigcup D^-_M(\Sigma) = M$.*

Using the definition of Cauchy surface, we can avoid therefore many inconveniences, *e.g.* temporal paradoxes, that will make complicated (or even impossible in some cases) the analysis of Dirac fields.

Definition 2.1.5. *Let M be a spacetime.*

- *We say that M is* ***globally hyperbolic*** *if and only if there exists a Cauchy surface Σ.*

- *Any open neighbourhood of a Cauchy surface Σ in M containing all causal curves for M whose endpoints lie in Ω will be called* ***globally hyperbolic open neighbourhood****.*

Notice that the restriction of M to Ω provides a globally hyperbolic spacetime $\Omega = (\Omega, g|_\Omega, \mathfrak{o}|_\Omega, \mathfrak{t}|_\Omega)$.

Remark 2.1.1. *This definition differs significantly from the original one of Leray in [Le52], but both are proved to be equivalent by Geroch in [Ge70].*

In this class of Lorentzian manifolds, we can find many notable examples: Minkowski, Friedmann-Lemaître-Robertson-Walker, Schwarzschild, Kerr, de Sitter and Rindler spacetime -for more details we refer to [Wa10]. Among all the globally hyperbolic spacetimes, the latter is of particular importance since it could be used as the arena for understanding what a quantum field theory in a curved spacetime ought to be.

Example 2.1. *A n+1-dimensional* ***Rindler spacetime*** $\mathscr{R}$ *is a Lorentzian manifold isometric to the subset of $n+1$-dimensional Minkowski spacetime $\mathscr{M} \equiv \mathbb{R}^{1,n}$*

$$\mathscr{R} = \left\{(t, x_1, \dots, x_n) \in \mathscr{M} \ \ with \ \ |t| < x_1\right\}$$

with the induced line element given by $ds^2 = dt^2 - \sum_{i=1}^n dx_i^2$. Replacing the coordinates t and x_1 by ρ and τ so that

$$t = \rho \sinh \tau \qquad x_1 = \rho \cosh \tau, \tag{2.1}$$

Rindler spacetime is covered by the coordinate range $(\tau, \rho) \in \mathbb{R} \times (0, \infty)$, $(x_2, \dots, x_n) \in \mathbb{R}^{n-1}$ and the line element becomes

$$ds^2 = \rho^2 d\tau^2 - d\rho^2 - \sum_{i=2}^n dx_i^2.$$

Surfaces of constant τ are Cauchy surfaces. Of course, this is not a curved spacetime, but its causal structure is nevertheless different from that of the whole Minkowski spacetime, e.g., it has incomplete geodesics. Note that, for any $\Delta \in \mathbb{R}$, the translations in the time coordinate τ,

$$\tau \mapsto \tau + \Delta, \qquad \rho \mapsto \rho, \tag{2.2}$$

describe a Killing symmetry. Hence it is an isometry of the Rindler spacetime. In fact, from equation (2.1)*, we see that this is nothing but a boost in the $t - x_1$ planes, whose orbits are the hyperbola of constant ρ asymptoting to the boundaries of the wedge as $\tau \to \pm\infty$. These orbits represent the world lines of uniformly accelerated observers with a proper acceleration ρ^{-1}.*

Example 2.2. *In this example we construct the so-called standard stationary spacetime. Let Σ be a n-dimensional manifold endowed with a Riemannian metric h, $f \in C^\infty(\Sigma)$ a smooth, strictly positive function and $w \in \Omega^\infty(\Sigma)$ a smooth one-form. We call* ***standard stationary spacetime*** *M the Cartesian product $\mathbb{R} \times \Sigma$, endowed it with the metric*

$$g := (\pi^* f)^2 dt^2 - \pi^* w \otimes dt - dt \otimes \pi^* w - \pi^* h,$$

where $\pi : M \to \Sigma$ and $t : M \to \mathbb{R}$ are the canonical projections. We can notice that g has Lorentzian signature and that the canonical vector field ∂_t on $\mathbb{R}$ gives rise to a Killing vector field ξ on M.

The spacetime M is globally hyperbolic if any of its timelike Killing vector fields is complete -we refer to [CFS08] for more details.

Theorem 2.1.1. *A spacetime M is globally hyperbolic if and only if*

- *There exists no closed causal curve in M and $J^+(p)_M \cap J^-(q)_M$ is either compact or empty for all $p, q \in M$;*
- *M is isometric to $\mathbb{R} \times \Sigma$ endowed with the line element $ds^2 = \beta dt^2 - h_t$, where β is a smooth and strictly positive function on $\mathbb{R} \times \Sigma$, h_t is a one-parameter family of smooth Riemannian metrics and for all $t \in \mathbb{R}$, $\{t\} \times \Sigma$ is an n-dimensional, spacelike, smooth Cauchy surface in M.*

Equivalence of the two conditions is the arrival point of an extended elaboration of the concept of global hyperbolicity. In particular, Bernal and Sánchez proved the first condition in [BS07], while the last one in [BS03]. Moreover, they also stated a significant structural result in [BS05], that we report in the following theorem.

Theorem 2.1.2. *Any globally hyperbolic Lorentzian manifold has a* ***Cauchy temporal function,*** *namely a smooth function $t : M \to \mathbb{R}$ with past-directed timelike gradient ∇t such that the levels $t^{-1}(s)$ are smooth spacelike Cauchy hypersurfaces if nonempty.*

This theorem plays a key role in the well-posedness of the Cauchy problem for a linear symmetric hyperbolic system as shown in Section 2.3. Since in the next section we will be interested in functions from a globally hyperbolic spacetime to a suitable vector space and in their support properties,we conclude this section with a useful definition:

Definition 2.1.6. *Let M be a (globally hyperbolic) spacetime and V a finite dimensional vector space. We denote*

(c) The space of ***smooth and compactly supported*** *V-valued functions on M with*

$$C^\infty_c(M,V) := \{f \in C^\infty(M,V) \,|\, \exists K \subset M \text{ compact, s.t. } \mathrm{supp}(f) \subset K\}$$

(sc) The space of ***smooth and spacelike compact*** *V-valued functions on M with*

$$C^\infty_{sc}(M,V) := \{f \in C^\infty(M,V) \,|\, \exists K \subset M \text{ compact, s.t. } \mathrm{supp}(f) \subset J_M(K)\}$$

(fc/pc) The space of ***smooth and future/past compact*** *V-valued functions on M with*

$$C^\infty_{fc/pc}(M,V) := \{f \in C^\infty(M,V) \,|\, \mathrm{supp}(f) \cap J^\pm_M(x) \text{ is compact } \forall x \in M\}$$

(tc) The space of ***smooth and timelike compact*** *V-valued functions on$M$ with*

$$C^\infty_{tc}(M,V) := C^\infty_{fc}(M,V) \cap C^\infty_{pc}(M,V).$$

2.2 Spin Geometry in a Nutshell

Before discussing how a spinor field and its covariant derivative are defined on globally hyperbolic spacetimes, we need an additional input.

Definition 2.2.1. *A **fibre bundle** over a spacetime M is a quadruple $E := (E, F, \pi_E, M)$ where:*

- ***The total space** E, **the typical fibre** F and **the base space** M are smooth manifolds ;*
- *$\pi : E \to M$ is a smooth surjective map ;*
- *$E_x := \pi^{-1}(x)$ is the fibre over $x \in M$.*

Furthermore we require that for every $x \in M$ there exists an open neighbourhood $U \subseteq M$ and a diffeomorphism $\varsigma : \pi^{-1}(U) \to U \times F$, such that the following diagram commutes

$$\begin{array}{ccc} \pi^{-1}[U] & \xrightarrow{\varsigma} & U \times F \\ {\scriptstyle \pi}\downarrow & \swarrow{\scriptstyle \mathrm{pr}_1} & \\ U & & \end{array}$$

*The pair (U, ς) fulfilling these conditions is called a **local trivialization** of E.*

Given a fibre bundle E, we denote the *space of smooth sections* as

$$\Gamma(E) := \{\sigma \in C^\infty(M, E) \,|\, \pi \circ \sigma = \mathrm{Id}_M\},$$

where $\mathrm{Id}_M : M \to M$ is the identity map. By generalizing Definition 2.1.6, the subscripts c, sc, fc/pc and tc shall refer to those sections whose support is compact, spacelike compact, future or past compact and timelike compact respectively.

Whenever all fibres and the typical fibre are finite dimensional vector spaces, and, for all $x \in M$, there exists a local bundle chart such that all the fibres are isomorphic to the typical fibre, we say that the fibre bundle is a *vector bundle* and we will denote it with the quadruple $VM := (VM, V, \pi, M)$.

If a fibre bundle P is equipped with a continuous right action R_G of a topological group G such that 1) G preserves the fibres of P, *i.e.* for every $y \in P_x$ then $yg \in P_x$ for all $g \in G$, 2) G acts freely and transitively on them, then we call $P = (P, F, R_G, \pi, M)$ *principal G-bundle* or simply *G-bundle*. This implies that each fibre of the bundle is homeomorphic to the group G itself. Now let P be a principal G-bundle over a manifold M and let Diff(V) denote the group of diffeomorphisms of another k-dimensional vector space V. Endow Diff(V) with the usual C^∞ topology. Then, to each continuous homomorphism $\varrho : G \to$ Diff(V), we construct a vector bundle over M with fibre V as follows. Consider the free left action of G on $P \times V$ given by

$$\varrho(\underline{x}, v) = (\underline{x}(g)^{-1}, \varrho(g)v)$$

for $\underline{x} \in P$, $g \in G$ and $v \in V$. Define $P \times_\varrho V$ to be the quotient space of this action. One can see that the projection $P \times V \to P \to M$ induces a projection map $\pi_\varrho : P \times_\varrho V \to M$. Hence $P \times V \to P$ is a vector bundle over M with fibre V and it is called the associated vector bundle.

We will not discuss the theory of fibre bundles and, for more details, refer to [Hu04] for a mathematical treatment and to [Na03] for an introduction motivated from physics. The only exceptions are the following definitions:

Definition 2.2.2. *Let VM be a vector bundle.*

- *We call **dual bundle** V^*M the vector bundle over M whose typical fibre over $x \in M$ is $(V_x^*M) = (V_xM)^*$, the dual vector space to V_xM .*
- *We say that a vector bundle VM is **(globally) trivial** if there exists a fibre preserving diffeomorphism from VM to the Cartesian product $M \times V$ restricting to a vector space isomorphism on each fibre.*

Notice that the space of smooth sections $\Gamma(VM)$ is an infinite-dimensional vector space and, whenever VM is trivial, it is isomorphic to $C^\infty(M,V)$.

We are now in the position to introduce the main geometric structures of this section which lies at the heart of the construction and of the analysis of the Dirac fields. Before starting, let us suggest [LM89] as a good reference for spin geometry and we also recommend the exposition in [Mi07, DHP09]. Our starting point is the following observation: While global Poincaré symmetry is not available in a generic curved spacetimes, even if globally hyperbolic, the proper, orthochronous Lorentz group $SO_0(1,n)$ is still a meaningful local symmetry group in agreement with Einstein equivalence principle. To encode the (local) Lorentz symmetry of a spacetime in a geometric object, we need the following definition.

Definition 2.2.3. *Given a vector bundle VM, we call **frame** ϵ over the point $x \in M$ the assignment of an ordered basis to the fibre V_xM, i.e. a map $p : \mathbb{K}^k \to V_xM$, being k the dimension of V and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.*

An application of this definition is the tangent bundle $TM = (TM, V, \pi, M)$ where $V = \mathbb{R}^{n+1}$, being $n+1 = \dim M$. We can have many different frames at every point, but they are related by a proper, orthochronous Lorentz transformation. If we consider all the oriented and time-oriented Lorentz frames over a point $x \in M$, we can gather all this information into a unique object.

Definition 2.2.4. *The **Lorentzian frame bundle** over a $n+1$-dimensional spacetime M is the principal bundle $\mathfrak{L} := (\mathfrak{L}, SO_0(1,n), R_{\mathfrak{L}}, \pi_{\mathfrak{L}}, M)$ with $SO_0(1,n)$ as typical fibre, right action $R_{\mathfrak{L}} : SO_0(1,n) \times \mathfrak{L} \to \mathfrak{L}$ which preserves the fibres and acts freely and transitively on them, and the projection map $\pi_{\mathfrak{L}} : \mathfrak{L} \to M$.*

To extend the definition of a spinor field on a generic spacetime, it seems reasonable to use the double covering group of $SO_0(1,n)$, namely, the identity component of the spin group $Spin(1,n)$.

Definition 2.2.5. *We call **spin group** $Spin(p,q)$ with $p,q \in \mathbb{N}$ the double cover of $SO(p,q)$.*

Therefore, any element of $Spin(p,q)$ induces an element of $SO(p,q)$. Such a surjective covering will be indicated as $\Theta : Spin(p,q) \to SO(p,q)$. For all $p,q > 0$, the spin group has two connected components and we denote the component connected to the identity with $Spin_0(q,p)$. To have a global notion of a spinor field, it is then necessary to make sure that the local double covering of $SO_0(1,n)$ can be consistently taken on the full spacetime M. If this is possible, we say that M has a spin structure.

Definition 2.2.6. *A **spin structure** on the Lorentzian frame bundle $\mathfrak{L}$ is a pair $(\mathfrak{S},\Xi)$ consisting of a principal bundle $\mathfrak{S} := (\mathfrak{S}, Spin_0(1,n), R_{\mathfrak{S}}, \pi_{\mathfrak{S}}, M)$ known as **spin bundle** and a map $\Xi : \mathfrak{S} \to \mathfrak{L}$ such that the following diagram is commutative:*

$$\begin{array}{ccccc} \mathfrak{S} \times Spin_0(1,n) & \xrightarrow{R_{\mathfrak{S}}} & \mathfrak{S} & \xrightarrow{\pi_{\mathfrak{S}}} & \\ \downarrow{\scriptstyle \Xi\times\Theta} & & \downarrow{\scriptstyle \Xi} & & \searrow \\ \mathfrak{L} \times SO_0(1,n) & \xrightarrow{R_{\mathfrak{L}}} & \mathfrak{L} & \xrightarrow{\pi_{\mathfrak{L}}} & M \end{array}$$

This definition should be read as follows: Two spin structures $(\mathfrak{S},\Xi)$ and $(\widetilde{\mathfrak{S}},\widetilde{\Xi})$ are equivalent if there exists a base point preserving bundle isomorphism $\varphi : \widetilde{\mathfrak{S}} \to \mathfrak{S}$ fulfilling $\Xi \circ \varphi = \widetilde{\Xi}$. A priori it may not be possible to construct a spin structure. To give an existence criterion, we first need to introduce the first two Stiefel-Whitney classes. Let $\{U_i\}_{i\in\mathbb{N}}$ be a simple open covering of M, which means that the intersection of any number of charts is either empty or contractible, and consider the transition function $\varrho_{ij} : U_i \cap U_j \to SO_0(1,n)$. We define the Čech 1-cochain $f(i,j)$ by

$$f(i,j) = \det(\varrho_{ij}) = \pm 1.$$

This is indeed an element of $C^1(M,\mathbb{Z}_2)$ since $f(i,j) = f(j,i)$. From the cocycle condition $\varrho_{ij}\varrho_{jk}\varrho_{ki} = \mathrm{Id}$, we verify that

$$\delta f(i,j,k) = \det(\varrho_{ij})\det(\varrho_{jk})\det(\varrho_{ki}) = \det(\varrho_{ij}\varrho_{jk}\varrho_{ki}) = 1.$$

Hence $f \in Z^1(M,\mathbb{Z}_2)$ and this defines an element $[f] := w_1(M) \in H^1(M,\mathbb{Z}_2)$ called *first Stiefel-Whitney class*. It can be shown that $w_1(M)$ is independent of the local frame chosen.

Theorem 2.2.1. *M is orientable if and only if $w_1(TM)$ is trivial.*

Now let us define a "lifting" $\widetilde{\varrho}_{ij} : U_i \cap U_j \to Spin(1,n)$ such that

$$\Theta(\widetilde{\varrho}_{ij}) = \varrho_{ij}, \qquad \widetilde{\varrho}_{ji} = \widetilde{\varrho}_{ij}^{-1}$$

where $\Theta : \mathfrak{S} \to \mathfrak{L}$ as in Definition 2.2.6. We want to remark that this lifting always exists locally. Taking into account

$$\Theta(\widetilde{\varrho}_{ij}\widetilde{\varrho}_{jk}\widetilde{\varrho}_{ki}) = \varrho_{ij}\varrho_{jk}\varrho_{ki} = 1$$

we have

$$\widetilde{\varrho}_{ij}\widetilde{\varrho}_{jk}\widetilde{\varrho}_{ki} \in \ker\Theta = \{\pm\mathrm{Id}\}$$

For $\widetilde{\varrho}_{ij}$ to define a spin bundle over M, they must satisfy the cocycle condition,

$$\widetilde{\varrho}_{ij}\widetilde{\varrho}_{jk}\widetilde{\varrho}_{ki} = \mathrm{Id}.$$

Let us define the Čech 2-cochain $f : U_i \cap U_j \to \mathbb{Z}_2$ by

$$\widetilde{\varrho}_{ij}\widetilde{\varrho}_{jk}\widetilde{\varrho}_{ki} = f(i,j,k)\mathrm{Id}.$$

It is easy to see that f is symmetric and closed. Thus f defines an element $w_2(M) \in H^2(M,\mathbb{Z}_2)$ called the *second Stiefel-Whitney class*. It can be shown that $w_2(M)$ is independent of the local frame chosen.

Theorem 2.2.2. *A manifold admits a spin structure if and only if its $w_2(M) = 0$.*

This condition was proven in the full generality by Borel and Hirzebruch in [BH59]. Instead, Geroch showed that this condition is automatically satisfied in every four-dimensional globally hyperbolic spacetime in [Ge68].

Corollary 2.2.1. *Every four-dimensional globally hyperbolic spacetimes M admits a spin structure and the spin bundle over M is trivial.*

In [Mi07], Milnor showed that there could be more than one spin structure for a given Riemannian manifold and that these different structures were labelled by elements of the group $H^1(M;\mathbb{Z}_2)$. In four-dimensional globally hyperbolic spacetimes the situation simplifies somewhat. On account of [Pa84], all orientable three-manifolds are parallelizable and since any four-dimensional globally hyperbolic spacetime M is isometric to the Cartesian product of $\mathbb{R}$ with an oriented 3-manifold, see Theorem 2.1.1, it follows that it is parallelizable. This is tantamount to finding a global section ϵ of the principal bundle $\mathfrak{L}$. In particular, this entails that $\mathfrak{L} \simeq M \times SO_0(1,3)$ via the principal bundle map $M \times SO_0(1,3) \ni (x,\lambda) \mapsto (\epsilon_a, \lambda) \in \mathfrak{L}$. Since both $\mathfrak{S}$ and $\mathfrak{L}$ are trivial for all four-dimensional globally hyperbolic spacetimes M, it follows the choice of the spin structure depends only on the map $\Xi : \mathfrak{S} \to \mathfrak{L}$ which reduces to choose a smooth $SO_0(1,3)-$valued function over M. In fact, all possible maps Ξ are of the form

$$\begin{aligned} \Xi : \mathfrak{S} \simeq M \times Spin_0(1,3) &\to M \times SO_0(1,3) \simeq \mathfrak{L} \\ (x,S) &\mapsto (x, f(x)\Theta(S)) \end{aligned}$$

with $\Theta : Spin_0(1,3) \to SO_0(1,3)$ and for some $f \in C^\infty(M, SO_0(1,3))$.

Remark 2.2.1. *Inequivalent spin-structure maps give rise to different spin connections.*

Once a spin structure $(\mathfrak{S},\Xi)$ has been chosen on a globally hyperbolic spacetime M , at a kinematic level, a spinor field could be defined as follows:

Definition 2.2.7. *Let M be a $n+1$-dimensional globally hyperbolic spacetime. A **spinor bundle** SM over M is the associated vector bundle to the spin bundle $\mathfrak{S}$. It takes the form*

$$SM = \mathfrak{S} \times_\varrho \mathbb{C}^N$$

*where ϱ is a faithful and unitary representation of the group $Spin_0(1,n)$ and $N = 2^{\lfloor \frac{n}{2} \rfloor}$, $\lfloor , \rfloor$ being the floor function. A **spinor field** ψ is a smooth section of SM. The **cospinor bundle** is the dual vector bundle*

$$S^*M = \bigsqcup_{x \in M} S^*_x M$$

*where $S^*_x M$ is the dual vector space to $S_x M$. A **cospinor field** ϕ is a smooth section of S^*M.*

At this stage, we can define spinors and cospinors to be sections of SM respectively of S^*M.

Remark 2.2.2. *Since a spin bundle $\mathfrak{S}$ over a four-dimensional, globally hyperbolic spacetime is trivial -see Corollary 2.2.1- then also the spinor bundle $SM = M \times \mathbb{C}^4$ is so. Therefore, a spinor field is nothing but a smooth function on M taking values in $\mathbb{C}^4$ and the same holds for the cospinor bundle and the cospinor fields.*

To write down the Dirac equation, one still needs γ-matrices and thus a Clifford algebra.

Definition 2.2.8. *Let V be a vector space over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $b : V \times V \to \mathbb{K}$ a quadratic form. We call **Clifford algebra**, the pair $C\ell(V) := (C\ell(V), j)$ where $C\ell(V)$ is an associative $\mathbb{K}$-algebra with the identity $\mathbf{1}$ and $j : V \to C\ell(V)$ is a linear map verifying*

$$j(v)^2 = b(v,v)\mathbf{1}$$

for all $v \in V$. If A is another $\mathbb{K}$-algebra with Id and $\widetilde{j} : V \to A$ a linear map satisfying $\widetilde{j}(v) = b(v)\mathbf{1}$, then there exists one and only one algebra homomorphism $\varphi : C\ell(V) \to A$ such that $\widetilde{j} = \varphi \circ j$.

Given a vector space V over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $b : V \times V \to \mathbb{K}$ be a quadratic form, we can construct a Clifford algebra as

$$C\ell(V) = \frac{T(V)}{\mathfrak{I}}$$

where $T(V) = \bigoplus_r V^{\otimes r}$ is the tensor algebra and $\mathfrak{I}$ is the ideal generated by the elements $e \in V$ such that

$$e \otimes v = b(v,v)\mathbf{1}.$$

This motivates the next proposition.

Proposition 2.2.1. *Let V be a vector space over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ endowed with a quadratic form b and $v_0, \ldots, v_n$ a basis of V such that*

$$b(v_\mu, v_\mu) = 0\,, \qquad \mu \neq \nu \in \{0, \ldots, n\}.$$

*Then the Clifford algebra $C\ell(V)$ is multiplicatively generated by the elements $e_0, \ldots, e_n \in V \subset C\ell(V)$ which satisfy the so called **Clifford relations***

$$v_\mu^2 = b(v_\mu, v_\mu)\mathbf{1}\,, \qquad \{v_\mu, v_\nu\} := v_\mu v_\nu + v_\nu v_\mu = 0, \quad \mu \neq \nu \in \{0, \ldots, n\}. \tag{2.3}$$

Let us now consider some explicit low-dimensional examples.

Example 2.3. *Consider* $\mathbb{R}^{1,1}$ *endowed with* η *=diag*$(1,-1)$ *and define the standard basis by* e_0, $e_1 \in \mathbb{R}^{1,1}$. *We embed linearly* $\mathbb{R}^{1,1}$ *into the space of* 2×2 *real matrices* $M(2,\mathbb{R})$ *via*

$$e_0 \mapsto \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad e_1 \mapsto \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.4}$$

As one verifies easily, these matrices satisfy the Clifford relations

$$\{\gamma_\mu,\gamma_\nu\} := \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}\boldsymbol{1}_2, \qquad \mu,\nu \in \{0,1\}.$$

The Clifford algebra $C\ell(\mathbb{R}^{1,1})$ *is isomorphic to the span of* $\boldsymbol{1}_2, \gamma_0, \gamma_1, \gamma_0\gamma_1 \in M(2,\mathbb{R})$*, the is the whole* $M(2,\mathbb{R})$.

Example 2.4. *As last example, we consider* $\mathbb{R}^{1,3}$. *One can find various different explicit representations of* $C\ell(\mathbb{R}^{1,3})$ *in the physics literature on quantum field theory. In the next chapter, we will use the so-called* ***chiral representation***

$$\gamma^0 = \begin{pmatrix} 0 & \boldsymbol{1}_2 \\ \boldsymbol{1}_2 & 0 \end{pmatrix}, \qquad \gamma^a = \begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix}, \quad a \in \{1,2,3\}$$

being σ^a *the Pauli matrices*

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The set of matrices $\{\gamma_\mu\}_{\mu=0,\dots,3}$ *satisfies the Clifford relations of* $\mathbb{R}^{1,3}$ *and one can show that the real subalgebra of* $M(4,\mathbb{C})$ *generated by these matrices is isomorphic to* $C\ell(\mathbb{R}^{1,3})$.

As we can already grasp from Example 2.3, it is a direct consequence of the Definition 2.2.8 that a basis for the Clifford algebra is given by the identity and by all products of γ-matrices, which entails that $\dim(C\ell(p,q)) = 2^{p+q}$. We wish to underline that $C\ell(\mathbb{R}^{p,q})$ is a $\mathbb{Z}_2$ -graded algebra: This arises if we introduce the automorphism $\alpha : C\ell(\mathbb{R}^{p,q}) \to C\ell(\mathbb{R}^{p,q})$ such that $\alpha(\gamma_\mu) = -\gamma_\mu$ for all possible μ. Since α^2 coincides with the identity map, we can always decompose:

$$C\ell(\mathbb{R}^{p,q}) = C\ell_0(\mathbb{R}^{p,q}) \oplus C\ell_1(\mathbb{R}^{p,q}),$$

where $C\ell_i(\mathbb{R}^{p,q}) = \{\gamma \in C\ell(p,q) | \alpha(\gamma) = (-)^i\gamma\}$. By direct inspection, one can realise that $C\ell_0(\mathbb{R}^{p,q})$ is the subalgebra of the full Clifford algebra generated by products of even numbers of γ_μ . On account of equation (2.3), the γ-matrices invertible, so the induced vector bundle maps are isomorphisms. In particular, one can introduce complex anti-linear vector bundle isomorphisms covering the identity which implement adjunction:

$$\begin{aligned} A : SM \to S^*M, &\qquad \psi \mapsto \psi^\dagger\gamma_0, \\ A^{-1} : S^*M \to SM, &\qquad \phi \mapsto \gamma_0\phi^\dagger, \end{aligned} \tag{2.5}$$

where $\dagger$ indicates the operations of transpose $(\cdot)^t$ and of conjugation $\overline{(\cdot)}$. We immediately use the adjunction map to introduce three pairings.

Definition 2.2.9. *Let SM a spinor over a spacetime M with a Cauchy surface Σ. We shall denote with*

- ***Spin product*** $\prec\ |\ \succ: S_xM \times S_xM \to \mathbb{C}$ *the paring defined by*

$$\prec \widetilde{\psi} \mid \psi \succ = ((A\psi)\widetilde{\psi})(x) \tag{2.6}$$

where A is the adjunction (2.5) *and $((A\psi)\widetilde{\psi})(x)$ is a fibrewise dual pairing of S_xM and S^*_xM obtained extending the dual pairing of $\mathbb{C}^N$ and $(\mathbb{C}^N)^*$;*

- ***Scalar product*** $(\ |\ )_\Sigma\, \Gamma_c(\Sigma, SM) \times \Gamma_c(\Sigma, SM) \to \mathbb{C}$ *the paring defined by*

$$\left(\psi \mid \widetilde{\psi}\right)_\Sigma = \int_\Sigma \prec \psi \mid \gamma^\mu n_\mu \widetilde{\psi} \succ d\Sigma, \tag{2.7}$$

where we have fixed an arbitrary Cauchy surface Σ with future pointing unit normal n;

- ***Spacetime inner product*** $\langle\ |\ \rangle : \Gamma_c(SM) \times \Gamma(SM) \to \mathbb{C}$ *the sesquilinear form defined by*

$$\left\langle \psi \mid \widetilde{\psi} \right\rangle := \int_M \prec \psi \mid \widetilde{\psi} \succ d\mu_g. \tag{2.8}$$

where $d\mu_g$ is the volume density of M.

The last ingredient that we need to define the Dirac operator is a parallel transport on the spinor bundle. The strategy is quite simple, namely: We 'pull back' the connection ω naturally defined on the frame bundle (since induced by the Lorentzian metric) along the map $\Theta : \mathfrak{S} \to \mathfrak{L}$, giving rise to a connection Ω on $\mathfrak{S}$ which may then be used to construct covariant derivatives of the spinor fields.

Definition 2.2.10. *Let $\omega : \mathfrak{L} \to T^*\mathfrak{L} \otimes \mathfrak{o}(p,q)$ denote the connection 1-form of the unique Levi-Civita connection on $\mathfrak{L}$. It induces the standard Levi-Civita connection on TM (and vice versa) which can be expressed as the covariant derivative*

$$\nabla : \Gamma(TM) \to \Gamma(TM \otimes T^*M), \quad \nabla \epsilon_a = \Gamma^b_{ac} \epsilon_b \otimes \epsilon^c$$

*where $\Gamma^b_{ac}$are the Christoffel symbols and ϵ is a Lorentzian frame. The pull-back $\Omega = (d\Theta)^{-1} \circ \Xi^* \circ \omega$ of ω to $\mathfrak{S}$, with $d\Theta : \mathfrak{spin}(p,q) \to \mathfrak{o}(p,q)$ denoting the derivative of the covering Θ at the identity, defines the **spin connection**, which by definition of SM as a bundle associated to $\mathfrak{S}$ can be specified as a **covariant derivative***

$$\nabla : \Gamma(SM) \to \Gamma(SM \otimes T^*M), \quad \nabla \widetilde{\epsilon}_A = \sigma^B_{aA} \epsilon^a \otimes \widetilde{\epsilon}_B,$$

where the spin connection coefficients are given by

$$\sigma^B_{aA} := \widetilde{\epsilon}^B((\Omega \circ \widetilde{\epsilon})[\widetilde{\epsilon}_* \circ \epsilon_a]\widetilde{\epsilon}_A)$$

and $\widetilde{\epsilon}_* : T^*M \to T^*\mathfrak{S}$ denotes the push-forward of $\widetilde{\epsilon}$ in the sense of cotangent vectors.

Using the γ-matrices together with the covariant derivatives both for spinors and for cospinors, we can introduce the first order linear differential operators

$$\nabla_{(s)} : \Gamma(SM) \to \Gamma(SM), \qquad \nabla_{(s)}\psi := \mathrm{Tr}_g(\gamma\nabla\psi)$$
$$\nabla_{(c)} : \Gamma(S^*M) \to \Gamma(S^*M), \qquad \nabla_{(c)}\phi := \mathrm{Tr}_g(\nabla\phi\gamma).$$

Here Tr_g denotes the metric-contraction of the covariant two-tensor $\gamma\nabla\psi$ taking values in $\Gamma(SM)$. Throughout this thesis the subscript (s) (resp. (c)) denotes spinor (resp. cospinor) quantities. A similar definition applies to $\nabla\phi\gamma \in \Gamma(S^*M)$. We have now all the necessary tools to introduce the Dirac operator and then we can single out the spinor and cospinor fields which are dynamically allowed in a curved background.

Definition 2.2.11. *We shall call dynamically allowed a* ***Dirac spinor*** $\psi \in \Gamma(SM)$ *which satisfies the* ***massive Dirac equation***

$$\mathscr{D}_m\psi := i\nabla_{(s)}\psi - m\psi = 0\,, \tag{2.9}$$

where $m \in C^\infty(M,\mathbb{R})$*. We call* ***formal adjoint of the Dirac operator***

$$\mathscr{D}_m^* : \Gamma(S^*M) \to \Gamma(S^*M),$$

unambiguously defined via

$$\langle \psi \mid A^{-1}\mathscr{D}_m^*\phi\rangle = \langle \mathscr{D}_m\psi \mid A^{-1}\phi\rangle$$

where $\langle\ |\ \rangle$ *is the spacetime inner product defined in* (2.8) *and* A *is the adjunction map* (2.5)*.* $\mathscr{D}_m^*$ *reads explicitly*

$$\mathscr{D}_m^*\phi = -i\nabla_{(c)}\phi - m\phi.$$

We call ***Dirac cospinor*** *every* $\phi \in \ker\mathscr{D}_m^*$*.*

Notice that we work in natural units ($\hbar = c = 1$) and in the rest of this thesis we will omit the subscript m when we will refer to the massless Dirac equation $\mathscr{D}\psi := i\nabla_{(s)}\psi = 0$. Characterizing the kernel of $\mathscr{D}_m$ and of $\mathscr{D}_m^*$ is the goal of the next section.

2.3 Linear Symmetric Hyperbolic Systems

In this section, we shall focus our attention on the class of globally hyperbolic spacetimes for a twofold reason: First of all they do not allow for pathological situations; secondly, they ensure the existence of a family of hypersurfaces on which initial data can be assigned. In this class of spacetimes, as we will see, the Cauchy problem for the massive Dirac equation is well posed, so we can characterize the space of solutions. Despite a possible way to proceed being to relate a solution of the massive Dirac equation to a solution of a hyperbolic differential equation as done in [BGP07, Wa12, Wr12], we prefer to use the theory of linear symmetric hyperbolic systems, following [Bä15]. These are a particular type of hyperbolic equations which contain the massive Dirac equation as shown in [Ni02]. We recommend [Le52, Fr82, Ta11] for more details and proofs.

Let $M \simeq \mathbb{R} \times \Sigma$ be a globally hyperbolic spacetime and let VM be a Hermitian vector bundle over M, *i.e.* a vector bundle equipped with a nondegenerate sesquilinear fibre metric $g(\cdot\,,\,\cdot)$ on each fibre V_x. Suppose that $P : \Gamma(V) \to \Gamma(V)$ is a linear first-order operator: In local coordinates, P takes the form

$$P = A^0(x)\partial_t + A^i(x)\partial_i + \mathscr{B}(x),$$

where t and x are coordinates for $\mathbb{R}$ and Σ respectively, the coefficients $A^0(x), A^i(x), \mathscr{B}(x)$ are $k \times k$-matrix functions of x and k is the dimension of the fibre. Its principal symbol $\sigma_P(x,k) : V_x \to V_x$ can be characterized by $P(fu) = fPu + \sigma_P(x,k)(df)u$, where $u \in \Sigma(V)$ and $f \in C^\infty(M,\mathbb{R})$. In local coordinates we have

$$\sigma_P(x,k) = A^0(x)k_t + A^i(x)k_i.$$

Definition 2.3.1. *Let $V \to M$ be a Hermitian vector bundle over a globally hyperbolic spacetime M. A linear differential operator $P : \Gamma(V) \to \Gamma(V)$ of first order is called a* ***linear symmetric hyperbolic system*** *over M if the following holds for every $x \in M$:*

(i) The principal symbol $\sigma_P(x,k) : V_x \to V_x$ is Hermitian with respect to $g(\cdot\,,\,\cdot)$ for every $k \in T^*M$;

(ii) For every future-directed timelike covector $k_0 \in T^*_x M$, the bilinear form $g(\sigma_P(x,k_t)\cdot\,,\,\cdot)$ on V_x is positive definite.

We can notice immediately that Definition 2.3.1, given by Bär in [Bä15] generalizes the one of Friedrichs in [Fr54]. In fact, in local coordinates, condition (i) is tantamount to requiring that the coefficients $A^i(x)$ are Hermitian matrices while condition (ii), choosing the covector $k_0 = dt$, implies that $\sigma_P(x,k_0) = A^0(x)$ is positive definite. Therefore the standard theory of existence, uniqueness, and smoothness of solutions of linear symmetric hyperbolic systems applies [Fr82] to P.

It is a remarkable fact that many equations in relativistic physics as well as most wave-type equations can be rewritten as a linear symmetric hyperbolic system. As an illustration, we explain now this reformulation for a two-dimensional Klein-Gordon equation.

Example 2.5. *Consider a scalar hyperbolic equation in $\mathbb{R}^{1,n}$ of the form*

$$\left(\partial_t^2 - \partial_x^2 + m^2\right)\phi = 0 \tag{2.10}$$

Introducing the vector v with $n+2$ components $v_1 = \partial_x\phi$, $v_2 = \partial_t\phi$, $v_3 = \phi$, equation (2.10) *reads*

$$\begin{pmatrix} \partial_t & -\partial_x & 0 \\ -\partial_x & \partial_t & -m^2 \\ 0 & 1 & \partial_t \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.11}$$

Hence, we can reduce equation (2.11) *to the linear symmetric hyperbolic system*

$$(A^0\partial_t + A^1\partial_x + \mathscr{B})v = 0,$$

with

$$A^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A^1 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathscr{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -m^2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Next, we want to deduce global information on the existence and uniqueness of solutions to the Cauchy problem. To achieve our goal, the following energy inequality is of fundamental importance.

Theorem 2.3.1. *Let M be globally hyperbolic, let P be a linear symmetric hyperbolic system over M and let $t : M \to \mathbb{R}$ be a Cauchy temporal function. For each $x \in M$ and each $t_0 \in t(M)$ there exists a constant $C > 0$ such that*

$$\int_{\Sigma^x_{t_1}} |u|_0^2 d\mu_{t_1} \leq C \int_{t_0}^{t_1} \int_{\Sigma^x_{t_0}} |Pu|_0^2 d\mu_s ds + \int_{\Sigma^x_{t_0}} |u|_0^2 d\mu_{t_0} e^{C(t_1 - t_0)}$$

holds for each $u \in C^\infty(V)$ and for all $t_1 \geq t_0$. Here $|\cdot|_0$ denote the norm corresponding to the scalar product $(\,|\,)_0 := g(\sigma_P(x,dt)\cdot\,,\,\cdot)$, $\Sigma_{t_s} := t^{-1}(s)$, $\Sigma^x_{t_s} := J^-(x) \cap \Sigma_{t_s}$ for $x \in M$, and $d\mu_{t_s}$ is the volume density of Σ_{t_s}.

A first consequence of the energy inequality concerns the maximal propagation speed for a solution of a linear symmetric hyperbolic system: We deduce that a "particle" ruled by such inequality can propagate at most with the speed of light.

Corollary 2.3.1. *Let M be globally hyperbolic, let $\Sigma \subset M$ be a smooth spacelike Cauchy hypersurface and let P be a linear symmetric hyperbolic system over M. Let $u \in \Gamma(E)$ and put $u_0 := u|_\Sigma$ and $f := Pu$. Then*

$$supp(u) \cap J^\pm(\Sigma) \subset J^\pm \cap \big((supp(f) \cap J^\pm(\Sigma) \cup supp(u_0)\big).$$

Moreover, we obtain uniqueness and existence of solutions for the Cauchy problem, where the last could also be achieved by gluing together local solutions. Uniqueness and existence proof for solutions to linear symmetric hyperbolic systems was also give in [FKT].

Corollary 2.3.2. *Let M be globally hyperbolic, let $\Sigma \subset M$ be a smooth spacelike Cauchy hypersurface and let P be a linear symmetric hyperbolic system over M. For any $f \in C^\infty(M,E)$ and $u_0 \in C^\infty(\Sigma,E)$ there is exactly one solution $u \in C^\infty(M,E)$ to the Cauchy problem*

$$\begin{cases} Pu = f, \\ u|_{t_0} = u_0. \end{cases}$$

Remark 2.3.1. *The unique solvability of the Cauchy problem allows us to introduce the time evolution operator as follows. For given initial data ψ_0 the Cauchy problem has a unique solution ψ. Evaluating this solution at some other time t, we obtain the operator $U_{t,t_0} : \psi_0 \mapsto \psi|_t$.*

Now that we have outlined the reason why the Cauchy problem for a linear symmetric hyperbolic system is well posed, we can characterize the space of solutions of the massive Dirac equation. Let us first consider a manifold M in the class of the stationary globally hyperbolic spacetime and let us write the Cauchy problem of the massive Dirac equation as

$$\begin{cases} i\partial_t \psi = H\psi, \\ \psi|_{t_0} = \psi_0. \end{cases} \tag{2.12}$$

The operator H was proved to be an elliptic, essentially self-adjoint operator by Chernoff in [Ch73]. This allows to express the solution of (2.12) using the spectral theorem for self-adjoint operators

$$\psi = e^{-itH}\psi_0 = \int_{\sigma(H)} e^{-i\omega t} dE_p \psi_0\,, \tag{2.13}$$

where dE_p and $\sigma(H)$ are respectively the spectral measure and the spectrum of H. In a recent paper [FiRö16a], this result was extended to the class Lorentzian spin manifolds with boundaries, in which every manifold satisfies the following conditions:

(i) The manifold is asymptotically flat at one asymptotic end;

(ii) There is a Killing vector field ξ which is tangential to the boundary and is timelike thereon;

(iii) The integral curves of the differential equation $\dot\gamma(t) = \xi\big(\gamma(t)\big)$ exist for all $t \in \mathbb{R}$;

(iv) There exists a spacelike hypersurface N with compact boundary ∂N with the property that every integral curve in (iii) intersects N exactly once.

Formula (2.13) was the starting point for a detailed analysis of the long-time behaviour of ψ using spectral methods and it was successfully used in [FKSY03, FiSm15, FiRö16b]. We can immediately notice that the operator $U := e^{-itH}$ realises a *time evolution operator*. The time evolution operator is a special case of the so-called *causal propagator*, that exists in every globally hyperbolic spacetime for linear symmetric hyperbolic PDEs.

Definition 2.3.2. *Let VM be a vector bundle over a globally hyperbolic manifold M and $P : \Gamma(VM) \to \Gamma(VM)$ be a linear differential operator. An* ***advanced/retarded Green's operator*** *of P is a linear map $E^{+/-} : \Gamma_c(VM) \to \Gamma_{sc}(VM)$ such that*

(i) $E^{+/-} \circ P = \mathrm{Id}_{\Gamma_c(VM)}$*;*

(ii) $P \circ E^{+/-} = \mathrm{Id}_{\Gamma_c(VM)}$ *;*

(iii) $\mathrm{supp}(E^{+/-}u_o) \subseteq J^{+/-}(\mathrm{supp}(u_o))$ *for all* $u_0 \in \Gamma_c(VM)$*;.*

If P and its formal adjoint P^* have advanced and retarded Green's operators then we say that P is ***Green hyperbolic****. In addition, we call* ***causal propagator*** *for P the operator $E := E^+ - E^- : \Gamma_c(VM) \to \Gamma_{sc}(VM)$.*

Theorem 2.3.2. *Let VM be a vector bundle over a globally hyperbolic spacetime M and $P : \Gamma(VM) \to \Gamma(VM)$ a linear symmetric hyperbolic system. Then P is Green hyperbolic and, denoting E be the causal propagator, we have the following exact sequence*

$$\{0\} \to \Gamma_c(VM) \xrightarrow{P} \Gamma_c(VM) \xrightarrow{E} \Gamma_{sc}(VM) \xrightarrow{P} \Gamma_{sc}(VM) \to \{0\}\,.$$

Since the massive Dirac equation (2.9) is a linear symmetric hyperbolic system, the space of smooth, spacelike compact solutions can be realised by

$$\mathrm{Sol}(\mathscr{D}_m) = E_m\big(\Gamma_c(SM)\big),$$

where E_m is the causal propagator for $\mathscr{D}_m$, and it enjoys the following isomorphism

$$\mathrm{Sol}(\mathscr{D}_m) \simeq \frac{\Gamma_c(SM)}{\mathscr{D}_m\Gamma_c(SM)}. \tag{2.14}$$

Taking the completion with respect to the scalar product (2.7), we obtain the Hilbert space $\mathscr{H}_{(s)}$. By duality, the space $\mathrm{Sol}(\mathscr{D}_m^*)$ of spacelike compact and smooth sections of S^*M such that $\mathscr{D}_m^*\phi = 0$ also enjoys the isomorphism

$$\mathrm{Sol}(\mathscr{D}_m^*) \simeq \frac{\Gamma_c(S^*M)}{\mathscr{D}_m^*\Gamma_c(S^*M)}. \tag{2.15}$$

Similarly we denote the Hilbert space obtained as

$$\mathscr{H}_{(c)} = \overline{\big(\mathrm{Sol}(\mathscr{D}_m^*), (\cdot \mid \cdot)_{(c)}\big)},$$

where $(\cdot \mid \cdot)_{(c)} := (A\,\cdot \mid A\,\cdot)_{(s)}$ with A is the adjunction map (2.5). Throughout this thesis, the subscripts (s) and (c) denotes spinor and cospinor quantities respectively.

Remark 2.3.2. *The scalar product* (2.7) *defined on* $\mathrm{Sol}(\mathscr{D}_m)$ *does not depend on the choice of* Σ *- we refer to [BD15] for the details. As a direct consequence, the time evolution operator* U_{t,t_0} *introduce Remark 2.3.1 is isometric. Thus by continuity, it extends uniquely to an isometry*

$$U_{t,t_0} : \mathscr{H}_{t_0} \to \mathscr{H}_t\,.$$

Since t_0 *can be chosen arbitrarily and the Cauchy problem can be solved forward and backward in time, this isometry is even a unitary operator. Moreover, these operators are a representation of the group* $(\mathbb{R};+)$.

Before concluding this section, we want to add a remark.

Remark 2.3.3. *Since the symbol of an operator depends only on the highest order terms, all the results above remain valid if the first-order operators are perturbed by the addition of arbitrary smooth, symmetric zero-order terms, e.g., "electromagnetic potentials" in the case of the Dirac operator.*

2.4 An Algebraic Approach to Quantum Dirac Fields

Having under control the dynamics of a classical Dirac fields, we are ready to quantise it. Our quantization scheme is based on the so-called algebraic approach to quantum field theory, initially developed by Haag and Kastler in Minkowski spacetime [HK64] and later extended to curved backgrounds by Dimock [Di80]. To introduce the algebraic approach to Dirac fields, we shall also profit from [Di82, Sa10b, BDH13, HS13, BDS14b, Za14, KM15, BDFY15, FrRe16].

In the algebraic approach to quantum field theory [Ha12, Mo13], the overall idea is the assignment of a suitable $*$-algebra to a physical system. Let us first recall the definition of a $*$-algebra and a $*$-morphism.

Definition 2.4.1. *An associative $\mathbb{C}$-algebra $\mathfrak{A}$ is called* ***$*$-algebra*** *if it admits an* ***involution****, namely, an anti-linear map $^* : \mathfrak{A} \to \mathfrak{A}$ such that $a^{**} = a$ and $(ab)^* = b^* a^*$ for any $a, b \in \mathfrak{A}$. Moreover, if $\mathfrak{A}$ is a Banach space with respect to a norm $\|\cdot\|$ which satisfies $\|ab\| \le \|a\|\|b\|$ and $\|a^* a\| = \|a^2\|$ for all $a, b \in \mathfrak{A}$ we called it* ***C^*-algebra****.*

Through this section, we shall only consider the so called unital $*$-algebras, *i.e.*, algebras in which there exists a multiplicative unit $1_{\mathfrak{A}} \in \mathfrak{A}$ satisfying $1_{\mathfrak{A}} a = a 1_{\mathfrak{A}} = a$.

Definition 2.4.2. *A $*$-morphism is a map $\psi : \mathfrak{A} \to \mathfrak{B}$ between $*$-algebras $\mathfrak{A}$ and $\mathfrak{B}$, which is an algebra morphism compatible with the involution, i.e. $\psi(a^*) = \psi(a)^*$ for all $a \in \mathfrak{A}$.*

Of course, not all $*$-algebras describe reality faithfully: As minimal mathematical requirements they should encode isotony, causality, and covariance.

Definition 2.4.3. *Let M denote a globally hyperbolic spacetime (M, g), let $\mathcal{O} \subseteq M$ be any contractible open bounded set and $\mathfrak{A}_{obs}(\mathcal{O})$ the associates $*$-algebra. We call* ***algebra of local observables*** *any algebra $\mathfrak{A}_{obs}(M)$ such that the following axioms are fulfilled:*

- ***Isotony:*** *if $\mathcal{O} \subseteq \mathcal{O}'$ then $\mathfrak{A}_{obs}(\mathcal{O}) \subseteq \mathfrak{A}_{obs}(\mathcal{O}')$.*
- ***Causality:*** *If $\mathcal{O} \cap J_M(\mathcal{O}') = \emptyset$, then $[a, b] = 0$ for all $a \in \mathfrak{A}_{obs}(\mathcal{O})$ and for all $b \in \mathfrak{A}_{obs}(\mathcal{O}')$, where the commutator is evaluated in the full algebra $\mathfrak{A}_{obs}(M)$;*
- ***Covariance:*** *For any isometry of M, i.e. a diffeomorphism $\iota : M \to M$ such that $\iota^* g = g$, there exists a $*$-isomorphism $\alpha_\iota : \mathfrak{A}_{obs}(M) \to \mathfrak{A}_{obs}(M)$ for which $\alpha_\iota(\mathfrak{A}_{obs}(\mathcal{O})) = \mathfrak{A}_{obs}(\iota(\mathcal{O}))$, for all open bounded sets $\mathcal{O} \subseteq M$.*

Finally the $*$-algebra of local observables $\mathfrak{A}_{obs}(M) := \bigcup_{\mathcal{O} \in M} \mathfrak{A}_{obs}(\mathcal{O})$.

In the algebraic approach, the algebra of observables is defined abstractly, seeking help from "generators" and "relations".

Definition 2.4.4. *Given a set G of generators, a $*$-algebra $\mathfrak{A}_G$ is said to be* ***freely generated by*** *G if there exists a map $\alpha : G \to \mathfrak{A}_G$ such that, for any other $*$-algebra $\mathfrak{B}$ and map $\beta : G \to \mathfrak{B}$, there exists a unique $*$-homomorphism $\varphi : \mathfrak{A}_G \to \mathfrak{B}$ such that $\beta = \varphi \circ \alpha$.*

Before discussing how to impose algebraic relations on the algebra $\mathfrak{A}_G$ freely generated by G, let us give an example. Consider an algebra $\mathfrak{A}_G$ freely generated by G in which is not present a unit, and suppose we want to add it. To this end, first we impose the relation "h-identity" $ha - ah = 0$ for all $a \in \mathfrak{A}_G$ and for a preferred element $h \in \mathfrak{A}_G$. Then we define $\mathfrak{A}_{G,\mathfrak{I}_h} := \mathfrak{A}_G / \mathfrak{I}_h$, where $\mathfrak{I}_h \subset \mathfrak{A}_G$ is the two-sided $*$-ideal generated by "h-identity", the set of finite linear combinations of products of $(ha - ah)$ for any $a \in \mathfrak{A}_G$. In case a set R of relations is imposed, one similarly takes the quotient with respect to all the $*$-ideals $\mathfrak{I}_R$ generated by each relation separately $\mathfrak{A}_{G,\mathfrak{I}_R} = \mathfrak{A}_G / \mathfrak{I}_R$.

Definition 2.4.5. *Given a $*$-algebra $\mathfrak{A}_G$ free on G and a set R whose elements are called relations, together with a map $\rho : R \to \mathfrak{A}_G$, a $*$-algebra $\mathfrak{A}_{G,R}$ is said to be* ***presented by the generators** G **and relations*** *R if there exists a $*$-homomorphism $r : \mathfrak{A}_G \to \mathfrak{A}_{G,R}$ such that, for any other $*$-algebra $\mathfrak{B}$ and maps $\beta : G \to \mathfrak{B}$ such that the composition of the relations with the canonical homomorphism $\varphi : \mathscr{A}_G \to \mathfrak{B}$ gives $\varphi \circ \rho = 0$, there exists a unique $*$-homomorphism $\varphi_R : \mathfrak{A}_{G,R} \to \mathfrak{B}$ such that $\varphi = \varphi_R \circ r$.*

An example of such $*$-algebra is the so-called Canonical Anti-commutation Relations algebra, which lies in the heart of the quantization of the Dirac fields.

Definition 2.4.6. *Let $\mathscr{H}$ be a complex Hilbert space and Υ be an antiunitary involution, i.e. an involution satisfying $(\Upsilon h_1, \Upsilon h_2) = (h_2, h_1)$ for all $h_1, h_2 \in \mathscr{H}$. A* ***CAR algebra*** *$\mathfrak{A}_{CAR}$ over $\mathscr{H}$ is a $*$-algebra generated by $B(h)$, $h \in \mathscr{H}$, its conjugate $B(h)^*$, $h \in \mathscr{H}$ and an $1_{\mathfrak{A}}$ identity which satisfy the following relations:*

(i) $B(h)$ *is $\mathbb{C}$-linear in h,*

(ii) $B(h)^* = B(\Upsilon h)$

(iii) $B(h_1)B(h_2)^* + B(h_2)^*B(h_1) = (h_1, h_2)1_{\mathfrak{A}_{CAR}}$.

Let us now summarise some important properties that play a key role in the formulation of the algebra of observables for Dirac fields. We refer to [BR97b, DG13] for more details.

Theorem 2.4.1. *Let $\mathscr{H}$ be a Hilbert space and $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two CAR-algebras over $\mathscr{H}$. It follows that there exists a unique $*$-isomorphism $\varphi : \mathfrak{A}_1 \to \mathfrak{A}_2$ such that $\varphi(B_1(h)) = B_2(h)$ for all $h \in \mathscr{H}$. Furthermore*

(i) $||B(h)||_{CAR} = ||h||_{\mathscr{H}}$ *, where $||\cdot||_{CAR}$ denotes the C^*-norm on the $\mathfrak{A}_{CAR}$ and $||\cdot||_{\mathscr{H}}$ on $\mathscr{H}$;*

(ii) $\mathfrak{A}_{CAR}$ *is simple, i.e. it has no closed two-sided $*$-ideals other than $\{0\}$ and the algebra itself;*

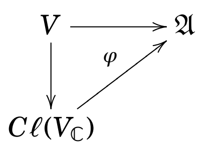
(iii) $\mathfrak{A}_{CAR}$ *is separable if, and only if, $\mathscr{H}$ is so;*

(iv) $\mathfrak{A}_{CAR}$ *is m-dimensional, with $m < \infty$, then $\mathfrak{A}_{CAR}$ is isomorphic with the C^*-algebra of $2^m \times 2^m$ complex matrices;*

(v) The algebras $\mathfrak{A}$ is $\mathbb{Z}_2$-graded, $\mathfrak{A}_{CAR} = \mathfrak{A}^{even}_{CAR} \oplus \mathfrak{A}^{odd}_{CAR}$ and $B(V) \subset \mathfrak{A}^{odd}_{CAR}$.

As an example, for any complex Hilbert vector space V, the Clifford algebra of the complexification $V_{\mathbb{C}}$ of V is a CAR algebra.

Corollary 2.4.1. *Let V be any complex Hilbert vector space and $\mathfrak{A}_{CAR}$ a CAR algebra over V. Then there exists a unique $*$-morphism $\varphi : C\ell(V_{\mathbb{C}}) \to \mathfrak{A}$ such that the following diagram commutes.*



We refer to [BG12] for more details. We are now in the position to construct the algebra of observables for the Dirac quantum fields. We refer to [Ar71, DHP09] for more details. Let M be an $n+1$-dimensional globally hyperbolic spacetime and suppose that it is possible to define a spin structure. In the following, we denote the space of pairs of spinorial test functions with

$$\mathfrak{D} := \Gamma_c(SM) \oplus \Gamma_c(S^*M)$$

and we equip it with the topology induced by the family of seminorms

$$|(f,g)|_{C^k} := \sup_{x\in\mathcal{M}} |\partial^k f(x)| + \sup_{y\in\mathcal{M}} |\partial^k g(y)|$$

(and $|.|$ is any norm on the spinors and co-spinors, respectively). Next we endow $\mathfrak{D}$ with the anti-linear involution map

$$\Upsilon : \mathfrak{D} \to \mathfrak{D}, \qquad \Upsilon(f \oplus g) := A^{-1}g \oplus Af,$$

(where A is the adjunction map (2.5)) and we introduce the scalar product

$$(f_1 \oplus g_1 \mid f_2 \oplus g_2)_{\mathfrak{D}} := (f_1 \mid f_2)_{(s)} + (g_1 \mid g_2)_{(c)},$$

for any $f_1, f_2 \in \Gamma_c(SM)$, $\phi_1, \phi_2 \in \Gamma_c(S^*M)$, where $(\ |\ )_{(s)}$ and $(\ |\ )_{(c)}$ are scalar products -see Section 2.2. Forming the completion, we thus obtain a Hilbert space $\mathscr{H}_{\mathfrak{D}}$.

Definition 2.4.7. *The C^*-**algebra of Dirac fields** $\mathfrak{F}(M)$ is defined as the quotient:*

$$\mathfrak{F}(M) := \frac{T(\mathscr{H}_{\mathfrak{D}})}{\mathfrak{I}}$$

where $T(\mathscr{H}_{\mathfrak{D}})$ is the tensor algebra built out of $\mathscr{H}_{\mathfrak{D}}$, while $\mathfrak{I}$ is the closed $*$-ideal which arises out of the relations

(i) $B(f \oplus g)^* = B(\Upsilon(f \oplus g)) = B(A^{-1}g \oplus Af)$

(ii) $\{B(f_1 \oplus g_1), B(f_2 \oplus g_2)^*\} = (f_1 \oplus g_1 \mid f_2 \oplus g_2)_{\mathfrak{D}}\, 1_{\mathfrak{A}_{CAR}}$,

(iii) $B(\mathscr{D}_m f \oplus \mathscr{D}_m^* g) = 0$,

for all $f \oplus g \in \mathscr{H}_{\mathfrak{D}}$, where $\mathscr{D}_m$ is the Dirac operator while $\mathscr{D}_m^*$ is its dual.

The algebra of Dirac fields $\mathfrak{F}(M)$ can be equipped with a natural topology induced from that on the tensor algebra. This is tantamount to the request that a sequence $h_j := \oplus_n h_{j,n} := \oplus_n (f_j \oplus g_j)_n$ is said to converge to h if and only if (i) every $h_{j,n} \to h_n$ in $\mathscr{H}_{\mathfrak{D}}^{\otimes n}$ with respect to the topology of uniform convergence of all derivatives on a fixed compact set, and (ii) it exists an $N \in \mathbb{N}$ such that $h_{j,n}$ vanishes for every $n > N$ and for every j. It is possible to recover the notion of spinor and cospinor quantum field starting from the B-generators as follows:

$$\Psi(g) := B(0 \oplus g) \qquad \text{and} \qquad \Psi^*(f) := B(f \oplus 0),$$

A short computation reveals

$$\begin{aligned}\{\Psi(g),\Psi^*(f)\} &= \{B(f\oplus 0),B(0\oplus g)\}=\{B(f\oplus 0),B\big(\Upsilon(0\oplus g)\big)^*\}\\ &=\big(f\oplus 0\,\big|\,\Upsilon(0\oplus g)\big)_{\mathfrak{D}}=\big(f\oplus 0\,\big|\,Ag\oplus 0\big)_{\mathfrak{D}}=\big(A^{-1}g\mid E_m f\big)_{(s)}=\langle A^{-1}g\mid E_m f\rangle\,,\end{aligned}$$

(where $\langle\mid\rangle$ is the spacetime inner product (2.8) and E_m is the causal propagator for $\mathscr{D}_m$), giving rise to the usual anti-commutation relations. Using relation (v) in Theorem 2.4.1, we split the algebra of Dirac fields in

$$\mathfrak{F}(M)=\mathfrak{F}^{\text{odd}}(M)\oplus\mathfrak{F}^{\text{even}}(M).$$

$\mathfrak{F}^{\text{even}}(M)$ can be understood as the subalgebra invariant under $B(f)\mapsto -B(f)$. The reason to choose such a subalgebra dwells in the fact that any two elements of $\mathfrak{F}^{\text{even}}(M)$ commute for spacelike separations.

Theorem 2.4.2. *Let M be a globally hyperbolic spacetime with a spin-structure and $\mathfrak{F}(M)$ be the algebra of Dirac fields on M . Then the following properties hold true:*

- ***Causality:*** *The elements of $\mathfrak{F}(M)$ localized in causally disjoint regions anti-commute. In particular the elements of $\mathfrak{F}^{even}(M)$ localized in causally disjoint regions commute.*
- ***Time-slice axiom:*** *Let $\Omega\subset M$ be a globally hyperbolic open neighbourhood of a spacelike Cauchy surface Σ for M. Then the map $\varphi:\mathfrak{F}(\Omega)\to\mathfrak{F}(M)$ is an isomorphism of C^*-algebras, where $\mathfrak{F}(M)$ and $\mathfrak{F}(\Omega)$ are the unital C^*-algebras of observables of Dirac fields respectively over M and over Ω*

Focusing our attention on $\mathfrak{F}^{\text{even}}(M)$, we have been able to ensure the requirement of *causality* - see Definition 2.4.3. In order to implement the *covariance*, we take into account only so-called "gauge invariant" elements of $\mathfrak{F}^{\text{even}}(M)$.

Definition 2.4.8. *We call **algebra of observables for Dirac fields** $\mathfrak{F}_{obs}(M)$ the subagebra of $\mathfrak{F}^{even}(M)$ assembled with the elements that are invariant under the action of $Spin_0(1,n)$. Such an action is defined by a straightforward extension of the known one on SM and on S^*M, first to $SM\oplus S^*M$ and subsequently to arbitrary outer tensor products of the latter.*

Now that we have defined the algebra of observables for the Dirac fields, we can discuss its representations. The key concept in this development is that of a state. A state on a $*$-algebra $\mathfrak{A}$ is nothing but a linear functional which take positive values on the positive elements of $\mathfrak{A}$ and they are of fundamental importance for the construction of representations.

Definition 2.4.9. *We call **(quantum) state** on a $*$-algebra $\mathfrak{A}$ a complex linear functional $\omega:\mathfrak{A}\to\mathbb{C}$, which is normalised, $\omega(1_{\mathfrak{A}})=1$, and positive, $\omega(a^*a)\geq 0$ for all $a\in\mathfrak{A}$.*

Notice that we have not demanded that the positive forms be continuous. For a C^*-algebra continuity is in fact a consequence of positivity -see [BR97a] for more details. The set of states over a $*$-algebra $\mathfrak{A}$ is a convex set: Given two states ω_1, ω_2 and $p\in[0,1]$, then also $\omega=p\omega_1+(1-p)\omega_2$ is a state too. With this in mind, we can formulate the following definition.

Definition 2.4.10. *A state ω is said to be* ***pure*** *(or extremal) if $\omega = p\omega_1 + (1-p)\omega_2$ is possible only if $\omega_1 = \omega_2 = \omega$.*

Due to the natural grading on $\mathfrak{F}(M)$, a state is specified once its n-points functions

$$\omega_n(f_1 \oplus g_1, \ldots, f_n \oplus g_n) := \omega\big(B(f_1 \oplus g_1) \otimes \ldots \otimes B(f_n \oplus g_n)\big) \qquad f_j \oplus g_j \in \mathfrak{D} \qquad j = 1, \ldots, n,$$

are assigned. This motives the next definition.

Definition 2.4.11. *We call* ***quasifree states*** *(or Gaussian states) those whose n-point functions vanish for odd n, while for even n, they are defined as*

$$\omega_n(f_1 \oplus g_1, \ldots, f_n \oplus g_n) = \sum_{\sigma \in S'_n} (-1)^{sign(\sigma)} \prod_{i=1}^{n/2} \omega_2\big(f_{\sigma(2i-1)} \oplus g_{\sigma(2i-1)}, f_{\sigma(2i)} \oplus g_{\sigma(2i)}\big),$$

where S'_n denotes the set of ordered permutations of n elements.

A characterisation of the quasifree states on a generic CAR algebra $\mathfrak{A}_{\mathrm{CAR}}$ was obtained by H. Araki in [Ar71]:

Lemma 2.4.1. *Let Υ be an involution on $\mathscr{H}$ and let $Q \in \mathscr{B}(\mathscr{H})$ satisfy:*

$$0 \le Q = Q^* \le 1, \qquad Q + \Upsilon Q \Upsilon = Id_{\mathscr{H}}. \tag{2.16}$$

Then

$$\omega\big(B(\Upsilon h_1)B(h_2)\big) = (h_1 \mid Q h_2) \qquad \forall h_1, h_2 \in \mathscr{H}, \tag{2.17}$$

defines a quasifree state on $\mathfrak{A}_{CAR}$. Conversely, for every quasifree states on $\mathfrak{A}_{CAR}$ there exists a bounded linear operator Q on $\mathscr{H}$ fulfilling (2.16) *and* (2.17).

Definition 2.4.12. *We call* ***basis projection*** *any projection operator Π on $\mathscr{H}$ satisfying conditions* (2.16).

In the rest of the paper, we denote the unique quasifree state of Lemma 2.4.1 with ω_Q. Since this characterisation is made for a generic CAR-algebra, let us extend Lemma 2.4.1 to the algebra of observables for Dirac fields.

Corollary 2.4.2. *Let Υ be an involution on the Hilbert space $\mathscr{H}_{\mathfrak{D}}$ and Π a orthonormal projector on the pre-Hilbert space $\Gamma_c(SM)$. Then the operator $P := \Pi \oplus (Id_{\mathscr{H}} - A\Pi A^{-1})$ is an orthonormal projector on $\mathscr{H}_{\mathfrak{D}}$ and satisfies $\Upsilon P \Upsilon + P = Id_{\mathscr{H}_{\mathfrak{D}}}$.*

After having defined different classes of quantum states and having derived a significant Lemmas that lies in the heart of the next chapters, we are in the position for giving the precise definition of representation.

Definition 2.4.13. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathscr{H}$ a complex Hilbert space. A* ***representation*** *of $\mathfrak{A}$ is defined to be a pair $(\mathscr{H}, \pi)$, where π is a $*$-morphism of $\mathfrak{A}$ into the space of bounded operators $\mathscr{B}(\mathscr{H})$. Moreover we say:*

- *The representation is* ***faithful*** *if* $\ker \pi = \{0\}$;
- *Given another representation* $\pi' : \mathfrak{A} \to \mathscr{B}(\mathscr{H})$, π *and* π' *are* ***unitarily equivalent*** *if there exists a surjective isometry* $U : \mathscr{H} \to \mathscr{H}$ *such that* $U\pi(a)U^{-1} = \pi'(a)$ *for any* $a \in \mathfrak{A}$;
- *A vector* $\Psi \in \mathscr{H}$ *is* ***cyclic*** *for* π *if* $\overline{\{\pi(a)\Psi \,|\, a \in \mathfrak{A}\}} = \mathscr{H}$.

Once a state has been fixed, the C^*-algebra can be represented in term of bounded linear operators on a Hilbert space. This is, indeed, a consequence of the renown GNS representation theorem for unital C^*-algebras, that ensures the existence of a representation of C^*-algebras.

Theorem 2.4.3. *Let* $\mathfrak{A}$ *be a* C^**-algebra with unit* $1_{\mathfrak{A}}$ *and* $\omega : \mathfrak{A} \to \mathbb{C}$ *a state. Then there exist a triple* $(\mathscr{H}_\omega, \pi_\omega, \Psi_\omega)$, *where* $\mathscr{H}_\omega$ *is a Hilbert space,* $\pi_\omega : \mathfrak{A} \to \mathscr{B}(\mathscr{H}_\omega)$ *a* $*$*-representation over* $\mathscr{H}_\omega$ *and* $\Psi_\omega \in \mathscr{H}_\omega$ *with* $||\Psi_\omega|| = 1$, *such that:*

(i) $\pi_\omega(\mathfrak{A})\Psi_\omega = \mathscr{H}_\omega$

(ii) $(\Psi_\omega, \pi_\omega(a)\Psi_\omega) = \omega(a)$ *for every* $a \in \mathfrak{A}$.

Moreover, If $(\mathscr{H}'_\omega, \pi'_\omega, \Psi'_\omega)$ *satisfies (i) and (ii), there exists a unitary operator* $U : H_\omega \to \mathscr{H}'_\omega$ *such that* $\Psi'_\omega = U\Psi'_\omega$ *and* $\pi'_\omega(a) = U\pi_\omega(a)U^{-1}$ *for any* $a \in \mathfrak{A}$.

We want to underline that the representation π_ω is continuous with respect to the operator norm $\|\cdot\|$ in $\mathscr{B}(\mathscr{H}_\omega)$, since $\pi_\omega(a) \in \mathscr{B}(\mathscr{H}_\omega)$ if $a \in \mathfrak{A}$, and also that $||\Psi_\omega||^2 = ||\omega|| = 1$.

Proposition 2.4.1. *Let* ω *be a state over the* C^**-algebra* $\mathfrak{A}$ *and let* $(\mathscr{H}_\omega, \pi_\omega, \Psi_\omega)$ *be the associated GNS representation. The following conditions are equivalent:*

(i) ω *is pure;*

(ii) The representation is weakly irreducible, i.e.

$$\begin{aligned}\pi'_\omega(\mathfrak{A}) := &\left\{T \in \mathscr{B}(\mathscr{H}) \,|\, (h_1 \mid T\pi(a)h_2) = (\pi(a)^\dagger h_1 \,|\, Th_2)\ \forall a \in \mathfrak{A},\ \forall h_1, h_2 \in \mathscr{H}\right\}\\ = &\{cI_{\mathscr{H}} : \mathscr{H} \to \mathscr{H} \,|\, c \in \mathbb{C}\}.\end{aligned}$$

Between all the possible representations, a central role in quantum field theory is played by the so-called Fock representations. In fact, they allow to describe particle creation and annihilation in terms of the Fock space.

Definition 2.4.14. *Let* Q *be an operator on* $\mathscr{H}$ *satisfying condition* (2.16). *The state* ω_Q *is called a* ***Fock state*** *and* π_Q *is called a* ***Fock representation***.

For more details concerning the physical interpretation of a Fock state we refer to [Fr15]. As shown in [Ar68], given two Fock representations it is always possible to find a Bogoliubov transformation that maps the first one in the second one.

Lemma 2.4.2. *For any two basis projections Π_1 and Π_2, there exists a **Bogoliubov transformation**, namely a unitary operator U on $\mathscr{H}$ commuting with Υ, such that $\Pi_1 = U\Pi_2 U^{-1}$.*

Example 2.6. *The time-evolution operator U introduced in the Remarks 2.3.1 and 2.3.2 is a Bogoliubov transformation as shown in [Se72].*

At a level of states, any Fock state can be obtained from another one by acting on it through an automorphism of the algebra.

Definition 2.4.15. *Let U be Bogoliubov transformation. We call **Bogoliubov $*$-automorphism** $\tau(U)$ of $\mathfrak{A}_{CAR}$ the automorphism given by $\tau(U)B(h) = B(Uh)$.*

The Bogoliubov automorphisms are fundamental to give a characterization of vacuum and KMS states in a curved background.

Definition 2.4.16. *Let τ_λ be a continuous one parameter group of automorphisms of a CAR-algebra $\mathfrak{A}_{CAR}$. A state ω of $\mathfrak{A}_{CAR}$ is said to be a **state of finite τ_λ energy** if there exists $\tilde{p}$ such that*

$$\int \omega\big(b(\tau_\lambda a)\big) f(\lambda) d\lambda = 0,$$

for $a, b \in \mathfrak{A}_{CAR}$ and whenever f is a Schwartz function satisfying

$$\hat{f}(p) = \int f(\lambda) e^{i\lambda p} d\lambda = 0$$

*for $p \geq \tilde{p}$. When $\tilde{p}$ can be chosen to be 0, ω is called τ_λ-**vacuum**. A state ω is called a **KMS state of τ_λ-inverse temperature** β if*

$$\int \omega(b\tau_\lambda a) f(\lambda) d\lambda = \int \omega\big((\tau_\lambda a) b\big) f(\lambda + i\beta) d\lambda$$

for $a, b \in \mathfrak{A}_{CAR}$ and $\hat{f} \in C_c^\infty(M)$ such that

$$f(\lambda) = \frac{1}{2\pi} \int \hat{f}(p) e^{-i\lambda p} dp.$$

Theorem 2.4.4. *Let $U(\lambda)$ be a continuous one parameter group of Bogoliubov transformations:*

$$U(\lambda) = \int_{\sigma(H)} e^{-i\lambda\omega} dE_\omega = e^{-itH}, \qquad H = \int_{\sigma(H)} \omega \, dE_\omega,$$

where dE_ω and $\sigma(H)$ are respectively the spectral measure and the spectrum of a self-adjoin operator H. Let χ_+ and χ_0 be two characteristic functions

$$\chi_+ = \begin{cases} 1 & \text{if } \lambda \in (0,\infty) \\ 0 & \text{otherwise} \end{cases} \qquad \chi_0 = \begin{cases} 1 & \text{if } \lambda = 0 \\ 0 & \text{otherwise}. \end{cases}$$

Then ω is a τ_λ-vacuum if and only if

$$\omega(ab) = \omega_{\chi_+}(a)\tilde{\omega}(b)$$

with $a \in A_{CAR}$ over $\mathscr{H} - \chi_0(H)\mathscr{H}$ and $b \in A_{CAR}$ over $\chi_0(H)\mathscr{H}$, where ω_{χ_+} is a Fock state and $\tilde{\omega}$ is an arbitrary state on A_{CAR} over $\chi_0(H)\mathscr{H}$. Moreover a KMS state of $\tau(U(\lambda))$ with inverse temperature β is unique and is given by a quasifree state ω_S for A_{CAR} with

$$S = (1 + e^{-\beta H})^{-1}.$$

Remark 2.4.1. *In a stationary spacetime, we can identify the operator H defined in Theorem 2.4.4 with the Dirac Hamiltonian. Moreover, the frequency splitting $\chi_+(H)$ is the unique vacuum state.*

Vacuum and KMS states can be further characterized as *passive states* [PW78]. Following the argument in [SV01], passivity allows proving that vacuum and KMS states satisfy the so-called Hadamard condition, which is regarded as being physically sensible within all possible states [FV13].

2.5 The Notion of Wavefront Set and the Hadamard Condition

Thanks to the seminal work of Radzikowski [Ra96a, Ra96b], the Hadamard condition has been translated into the language of microlocal analysis, as a constraint on the wavefront set of the bidistribution associated to the two-point function of the state. This approach was rapidly extended to the case of Dirac fields [SV01, Ho01, DHP11], to gauge fields [SDH14, DS13, WZ14, GW15], to the quantization of linearized gravity [BFR16, FMR16a] and even to supersymmetric quantum field theories [DGMS17]. We start by defining conical neighbourhoods and regular directed points.

Definition 2.5.1. *A **conical neighbourhood** of a point $k \in \mathbb{R}^n \setminus \{0\}$ is a set $\Gamma \subset \mathbb{R}^n$ such that Γ contains the ball $B(\widetilde{k},\varepsilon) = \{\widetilde{k} \in \mathbb{R}^n : |\widetilde{k} - k| \leq \varepsilon\}$ for some $\varepsilon > 0$ and, for any $\widetilde{k} \in \Gamma$ and any $\lambda > 0$, $\lambda\widetilde{k}$ belongs to Γ .*

Definition 2.5.2. *One calls $(x,k) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ a **regular directed point** for a distribution $u \in D'(\mathbb{R}^n)$ if there exists a function $f \in C_c^\infty(\mathbb{R}^n)$ with $f(x) = 1$, and a conical open neighbourhood Γ of k in $\mathbb{R}^n \setminus \{0\}$ such that*

$$\sup_{\widetilde{k}\in\Gamma}(1 + |\widetilde{k}|)^N |\widehat{fu}(\widetilde{k})| \leq C_N$$

holds for all $N \in \mathbb{N}$, where $\widehat{fu}$ denotes the Fourier transform of the distribution fu.

The relevance of the concept of regular directed points also stems from the following theorem [Hö07].

Theorem 2.5.1. *A distribution $u \in D'(\mathbb{R}^n)$ is a smooth function if and only if $\widehat{u}$ is **fast decreasing**, namely for any integer N , there exists a constant C_N such that $|\widehat{u}(k)| \leq C_N(1 + |k|)^{-N}$ for all $k \in \mathbb{R}^n$.*

Theorem 2.5.1 implies that any singularity of a distribution u can be detected by an absence of a fast decrease in a direction: A point x is in the singular support if and only if there exists a direction k where the Fourier transform is not fast decreasing. However, if $x \in$ sing supp u, there can be directions k such that (x,k) is regular directed. In Example 2.8, we shall see that $\widehat{fu}(k)$ is rapidly decreasing for $k > 0$ but not for $k < 0$. This leads us to the definition of wavefront set.

Definition 2.5.3. *The **wavefront set** $\mathrm{WF}(u)$ is defined as the complement in $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ of the set of all regular directed points for u.*

As we already anticipated, $\mathrm{WF}(u)$ consists of pairs (x,k) for which the Fourier transform of $\widehat{fu}$ is not rapidly decaying along the direction k for large $|k|$, no matter how closely f is concentrated around x.

Example 2.7. *Let us calculate the wavefront set of the Dirac δ-distribution at* 0 *in* $\mathbb{R}$. *First, the singular support of the distribution is* $\{0\}$ *and therefore we need to concentrate only on the set of singular directions at 0. For any* $f \in C_c^\infty(\mathbb{R}^n)$, *we get*

$$\widehat{f\delta} = \frac{1}{2\pi} f(0),$$

which is not decaying in any direction. Therefore,

$$\mathrm{WF}(\delta) = \{x\} \times \mathbb{R} \setminus \{0\}.$$

Example 2.8. *Another example is the distribution*

$$u = \lim_{\varepsilon \to +0} \frac{1}{x + i\varepsilon}$$

defined by

$$fu := \lim_{\varepsilon \to +0} \int_{\mathbb{R}} dx \frac{f(x)}{x + i\varepsilon}.$$

The Fourier transform of u *for* $\varepsilon > 0$ *can be calculated easily using the residue theorem*

$$\widehat{u}(k) = \lim_{\varepsilon \to +0} \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-ikx}}{x + i\varepsilon} dx = \lim_{\varepsilon \to +0} -iH(k)e^{-\varepsilon k} = -iH(k),$$

where H *is the Heaviside function. Moreover, the Fourier transform of* fu *can be evaluated directly*

$$\widehat{fu}(k) = \frac{1}{2\pi}(\widehat{f} * \widehat{u})(k) = \frac{-i}{2\pi} \int_{\mathbb{R}} f(\eta) H(k - \eta) d\eta = \frac{-i}{2\pi} \int_{-\infty}^{k} f(\eta) d\eta,$$

which decays rapidly as $k \to \infty$ *and tends to* $f(0)$ *as* $k \to -\infty$. *Therefore,*

$$\mathrm{WF}(u) = \{0\} \times \mathbb{R}^+.$$

Theorem 2.5.2. *Let* $u \in D'(U)$, $U \subset \mathbb{R}^n$ *open and non-empty.*

(i) $\mathrm{WF}(u)$ *is empty if and only if* u *is smooth.*

(ii) For any collection of finitely many $u_1, \ldots, u_n \in D'(U)$ *we have*

$$\mathrm{WF}(\sum_j u_j) \subseteq \bigcup_j \mathrm{WF}(u_j). \tag{2.18}$$

(iii) If P *is a partial differential operator with smooth coefficients:*

$$\mathrm{WF}(Pu) \subseteq \mathrm{WF}(u). \tag{2.19}$$

(iv) Let $V \subset \mathbb{R}^n$ *be an open set and let* $\varrho : U \to V$ *be a diffeomorphism. The pull-back* $\varrho^* u \in D'(U)$ *of* u *defined by* $\varrho^* u(f) = u(\varrho_* f)$ *for all* $f \in C_c^\infty(V)$ *fulfills*

$$\mathrm{WF}(\varrho^* u) = \varrho^*(\mathrm{WF}(u)) := \{(\varrho^{-1}(x), \varrho^* k) \mid (x,k) \in \mathrm{WF}(u)\}$$

where ϱ^* *denotes the pull-back of* ϱ *in the sense of cotangent vectors.*

(v) Let P be a normally hyperbolic operator whose principal symbol is real valued. If $u,v \in D'(U)$ are such that $Pu = v$ then:

(a) WF(u) $\subseteq$ char(P)$\cup$WF(v), where char(P) $:= \{(x,k) \in T^*U \setminus 0 \mid g^{\mu\nu}k_\mu k_\nu = 0\}$;

(b) WF(u)$\setminus$WF(v) is invariant under the local flow of σ_P on $T^*U\setminus$WF(v), where σ_P is the principal symbol of P.

Property (v) goes under the name of *propagation of the singularities*. From (iv) we conclude that the wavefront set transforms covariantly under diffeomorphisms as a subset of T^*U, with U an open subset of $\mathbb{R}^n$. Therefore we can immediately extend the definition of WF to distributions on a manifold M simply by patching together wavefront sets in different coordinate patches of M with the help of a partition of unity.

Definition 2.5.4 (WF on a manifold)**.** *Let $\varrho : M \to \mathbb{R}^n$ be a local diffeomorphism around a point $x \in M$ and $u \in D'(M)$. We define* $\mathrm{WF}(u)$ *by saying that* $(x,k) \in \mathrm{WF}(u)$ *if and only if* $(\varrho x, \varrho_* k) \in \mathrm{WF}((\varrho^{-1})^* u)$.

Owing to the transformation properties of the wavefront set under local diffeomorphisms one can see that this definition is independent of the choice of the chart ϱ, and moreover, $\mathrm{WF}(u)$ is a subset of $T^*M \setminus \{0\}$, the cotangent bundle with the zero section removed.

We are now in the position to extend the definition of wavefront set for vector valued distributions. Let E be a smooth vector bundle over a $n+1$-dimensional manifold M with typical fibre $\mathbb{C}^k$ and bundle projection π_E . Let $U \subset M$ be an open subset and let ϵ be a local frame of E over U. Such a local trivialization induces a one-to-one correspondence between $C_c^\infty(E_U)$ and $\bigoplus_k C_c^\infty(U)$ by assigning to each $f \in C_c^\infty(E_U)$ the $(f_1,\ldots,f_k) \in \oplus_k \mathbb{C}_c^\infty(U)$ with $f^a \epsilon_a = f$. In turn, this induces a one-to-one correspondence between $\left(\Gamma_c(E_U)\right)'$ and $\bigoplus_r D'(U)$, via mapping $u \in \left(C_c^\infty(E_U)\right)'$ to $(u_1,\ldots,u_k) \in \bigoplus_r D'(U)$ given by $u_a(h) = u(h\epsilon_a)$, where $h \in D(U)$. With this notation, one defines for $u \in \left(C_c^\infty(E_U)\right)'$ the wavefront set as

$$\mathrm{WF}(u) := \bigcup_{a=1}^{r} \mathrm{WF}(u_a),$$

i.e. the wavefront set of u is defined as the union of the wavefront sets of the scalar component-distributions in any local trivialization over U. Using (2.18) and (2.19) it is straightforward to see that this definition is independent of the choice of local trivializations.

Definition 2.5.5 (WF on a vector bundle)**.** *Let $u \in \left(\Gamma_c(E)\right)'$, $(x,k_x) \in T^*M \setminus \{0\}$. Then (x,k_x) is defined to be in* $\mathrm{WF}(u)$ *if, for any neighbourhood U of x over which E trivializes, (x,k_x) is in* $\mathrm{WF}(u_U)$ *where u_U is the restriction of u to $\Gamma_c(E)$.*

The properties of $\mathrm{WF}(u)$ are similar to those in the case of scalar distributions; obviously (2.18) and (2.19) generalize to the vector bundle case. Now we have all the tools to formulate the Hadamard condition for vector valued bidistributions.

Definition 2.5.6. *A bidistribution $u \in \left(\Gamma_c(E) \times \Gamma_c(E)\right)'$ satisfies the **Hadamard condition** if and only if*

$$\mathrm{WF}(u) = \{(x, y, k_x, -k_y) \in T^*M^2 \setminus \{0\} | \, (x, k_x) \sim (y, k_y), \; k_x \rhd 0\},$$

where $(x, k_x) \sim (y, k_y)$ means that there is a null geodesic γ connecting x to y, such that k_x is cotangent to γ at x and k_y is the coparallel transport along γ of k_x from x to y. In addition, $k_x \rhd 0$ selects future pointing covectors.

Notice that we can apply the definition of the Hadamard condition to quasifree states. On account of the isomorphisms (2.14) and (2.15), we can associate uniquely to ω_2 a bidistribution in $\left(\Gamma_c\left(SM \oplus S^*M\right)^{\otimes 2}\right)'$ by means of the relation

$$\widetilde{\omega_2}(u \oplus v, u' \oplus v') := \omega_2(\psi_u \oplus \phi_v, \psi_{u'} \oplus \phi_{v'}),$$

where $\psi_{u^{(\prime)}} := E_m u^{(\prime)}$, $\phi_{v^{(\prime)}} := E^*_m v^{(\prime)}$ where $u, u' \in \Gamma_c(SM)$, $v, v' \in \Gamma_c(S^*M)$ while E_m, E^*_m are the causal propagator for $\mathscr{D}_m$, $\mathscr{D}^*_m$. Whenever a bidistribution associated to a quasifree state satisfies the Hadamard condition, we denote it as *Hadamard state*. However, in [Sa10a] it has been shown that this condition is sufficient also for non-Gaussian states, because the singularities of all n-point functions are already determined by the singularities of $\widetilde{\omega_2}$ and by the canonical anticommutation relations:

$$\omega^+(f, g) + \omega^-(f, g) := \widetilde{\omega_2}(0 \oplus g, f \oplus 0) + \widetilde{\omega_2}(f \oplus 0, 0 \oplus g) = \left\langle A^{-1} g \mid E_m f \right\rangle, \tag{2.20}$$

where A is the adjuction map and E_m is the causal propagator for the Dirac operator.

2.5.1 On the Local Hadamard form

Let us consider a geodesically convex neighbourhood O of x_0, i.e. a subset of M such that any points can be connected by a unique geodesic, and let $T_x O$ be a subset of $T_x M$ such that the exponential map $\exp_x : T_x O \to O$ is well-defined for all $x \in O$. Now let t be a time function on M and

$$\sigma_{\pm\epsilon}(x, y) = \sigma(x, y) \pm 2i\epsilon(t(x) - t(y)) + \epsilon^2 = \frac{1}{2} g(\exp_x^{-1}(x), \exp_x^{-1}(y)) \pm 2i\epsilon(t(x) - t(y)) + \epsilon^2 .$$

Definition 2.5.7. *We say that ω is of* ***local Hadamard form*** *if, for every $x_0 \in M$ there exists a geodesically convex neighbourhood O such that $\omega^\pm(x, y)$ on $O \times O$ takes the form*

$$\begin{aligned}\omega^\pm(x, y) &= \pm \frac{1}{8\pi^2} \mathscr{D}^*_y \left(\frac{U(x, y)}{\sigma_{\pm\epsilon}(x, y)} + V(x, y) \log\left(\frac{\sigma_{\pm\epsilon}(x, y)}{\lambda^2}\right) + W(x, y)\right) \\ &= \pm \frac{1}{8\pi^2} \mathscr{D}^*_y \left(H^\pm(x, y) + W(x, y)\right),\end{aligned}$$

where λ is an arbitrary length scale. Here U, V, and W are smooth bispinors and V, and W can be expanded in powers of σ

$$V(x, y) = \sum_{n=0} V_x(x, y)\sigma(x, y)^n , \qquad W(x, y) = \sum_{n=0} W_x(x, y)\sigma(x, y)^n .$$

We furthermore require

$$\begin{cases} \mathscr{D}^*_x \mathscr{D}_x H^\pm(x, y) \in \Gamma(SM \otimes S^*M) \\ \mathscr{D}^*_y \mathscr{D}_y H^\pm(x, y) \in \Gamma(SM \otimes S^*M) \end{cases} \quad \text{and} \quad H^+(f, g) - H^-(f, g) = i \langle f \mid Eg \rangle \tag{2.21}$$

where $f \in \Gamma_c(SM)$ and $g \in \Gamma_c(S^*M)$.

Note that the above series expansion of V does not necessarily converge on general smooth spacetimes, however, it is known to converge on analytic spacetimes [Ga64]. Let us remark that Equation (2.21) determines the values of the coefficient U, V and W. By direct computation, U and V turn out to depend only on the geometry and on the mass m of the Dirac fields, while the positivity of the state is encoded in W. This property can be shown to be "functorial" (using a categorical language) and this allows to define a locally covariant definition of normal ordering. For more details we refer to [KW91, Ha10, DNP16]. An advantage of using the local Hadamard form lies in the following formulation catching the essence of the Hadamard condition:

Theorem 2.5.3. *A state ω on the algebra of Dirac fields $\mathfrak{F}(M)$ is Hadamard if and only if*

$$\omega_2(f_1 \oplus g_1, f_2 \oplus g_2) - \mathfrak{h}(f_1 \oplus g_1, f_2 \oplus g_2)$$

has a smooth integral kernel, where $\mathfrak{h}(f_1 \oplus g_1, f_2 \oplus g_2) = H^+(f_2, \mathscr{D} g_1) - H^-(f_1, \mathscr{D} g_2)$.

Before closing this section and the chapter we give few examples of Hadamard states.

- First of all, the deformation argument of Fulling, Narcowich and Wald [FSW78, FNW81] shows that Hadamard states for free fields exist on any globally hyperbolic spacetime. Employing deformation techniques and adiabatic limits, we also recall the work in [FiSt15, DD16, DG16]
- All vacuum states and thermal equilibrium states on static spacetimes are Hadamard states [SV00].
- The Hartle-Hawking-Israel state for a free quantum Klein-Gordon field on a spacetime with a static, bifurcate Killing horizon and a wedge reflection [Sa15, Gé16].
- The Bunch-Davies state on de Sitter spacetime is a Hadamard state [Al85]. It has been shown in [DMP09a, DMP09b] that this result can be generalized to asymptotically de Sitter spacetimes.
- Holographic arguments have been used in [DMP06] to construct distinguished Hadamard states on asymptotically flat spacetimes [DS13, BDM14], to rigorously construct the Unruh state in Schwarzschild spacetimes [DMP11], to construct asymptotic vacuum and thermal equilibrium states in certain classes of Friedmann-Robertson-Walker spacetimes [DHP11].
- In [GW14, GW16a, GW15, GW16b, GW16c], it has been outlined a new framework aimed at the construction of Hadamard states via methods proper of pseudo-differential calculus.
- We conclude reminding two recent analytic constructions developed in [FiRe16, DM16] and successful applied to [FMR16a, FiRe17]. These techniques represent the topic of Chapter 4.

In the next chapter, we outline a technique similar to the one employed in [FiRe15], in order to investigate the construction of quantum states in Rindler spacetime [FMR16b].

CHAPTER 3

FOCK STATES IN RINDLER SPACETIME

In [Fu73] it was shown that in a quantum field theory on curved spacetime, the interpretation of physical states in terms of particles and anti-particles depends on the observer. This becomes most apparent in the renown Unruh effect [Un76], showing that in the usual vacuum state in Minkowski space, a uniformly accelerated observer detects particles and anti-particles in a thermal state. In mathematical terms, the observer-dependence of the particle interpretation is reflected in the freedom to choose a ground state. In this chapter, we present a novel way of looking at these issues. Starting from the restriction of the standard Minkowski-space theory to the Rindler wedge, we construct first a self-adjoint operator, the *fermionic signature operator*, and later, with the help of the spectral calculus and Lemma 2.4.1, the Fulling-Rindler and the Unruh state.
The results of this chapter have already appeared as preprint in [FMR16b].

3.1 Embedding in Minkowski Spacetime

As we already saw in Example 2.1, the 2-dimensional Rindler wedge $\mathscr{R}$ could be seen as a globally hyperbolic subspace of the two-dimensional Minkowski spacetime $\mathscr{M}$, namely

$$\mathscr{R} = \big\{(t,x) \in \mathscr{M} \;\; \text{with} \;\; |t| < x\big\}$$

with the induced line element $ds^2 = dt^2 - dx^2$. Since $\mathscr{M}$ and $\mathscr{R}$ are contractible, notice that the spin structure is uniquely defined and the spinor bundles over $\mathscr{M}$ and $\mathscr{R}$ are trivial and satisfy

$$S\mathscr{R} = \mathscr{R} \times \mathbb{C}^2 \subset \mathscr{M} \times \mathbb{C}^2 \mathscr{M} = S\mathscr{M}\,.$$

In this setting, the *spin product* (2.6) takes the form

$$\prec \cdot \,|\, \cdot \succ = \left\langle \cdot, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \right\rangle_{\mathbb{C}^2}, \tag{3.1}$$

where $\langle \cdot | \cdot \rangle_{\mathbb{C}^2}$ is the canonical scalar product on $\mathbb{C}^2$. The *Dirac operator* reads

$$\mathscr{D}_m = i\gamma^0 \partial_t + i\gamma^1 \partial_x - \mathbf{1}_{\mathbb{C}^2} m \tag{3.2}$$

where $m \in (0,\infty)$ is the rest mass and γ^0, γ^1 are the matrices constructed in Example 2.3. Taking smooth and compactly supported initial data on suitable Cauchy surfaces and solving the Cauchy problem, one obtains compactly supported solution to the massive Dirac equation in both Rindler and Minkowski spacetime.

Definition 3.1.1. *We call* ***space of Dirac solution in Rindler spacetime*** *the Hilbert space* $\mathscr{H}_{\mathfrak{R}}$ *obtained taking the completion of*

$$\mathrm{Sol}(\mathscr{R}) := \{\psi \in C^\infty_{sc}(\mathscr{R}, \mathbb{C}^2) \,|\, \mathscr{D}_m \psi = 0\}$$

with respect to the scalar product

$$(\cdot \,|\, \cdot)_{\mathfrak{R}} := 2\pi \int_0^\infty \prec \cdot \,|\, \gamma^0 \cdot \succ |_{(t=0,x)}\, dx\,. \tag{3.3}$$

Similarly, we call ***space of Dirac solution in Minkowski spacetime*** *the Hilbert space* $\mathscr{H}_{\mathfrak{M}}$ *obtained taking the completion of*

$$\mathrm{Sol}(\mathscr{M}) := \{\Psi \in C^\infty_{sc}(\mathscr{M}, \mathbb{C}^2) \,|\, \mathscr{D}_m \Psi = 0\}$$

with to respect the scalar product

$$(\cdot \,|\, \cdot)_{\mathfrak{M}} := 2\pi \int_{-\infty}^\infty \prec \cdot \,|\, \gamma^0 \cdot \succ |_{(t=0,x)}\, dx\,. \tag{3.4}$$

We denote the norm on these Hilbert spaces by $\|\cdot\|_{\mathfrak{R}} := \sqrt{(\cdot \,|\, \cdot)_{\mathfrak{R}}}$ *and* $\|\cdot\|_{\mathfrak{M}} := \sqrt{(\cdot \,|\, \cdot)_{\mathfrak{M}}}$.

To avoid confusion, we denote consistently wave functions in Minkowski space by capital Greek letters, whereas wave functions in Rindler spacetime are denoted by small Greek letters.

Lemma 3.1.1. *Let* $\mathscr{H}_{\mathfrak{R}}$ *and* $\mathscr{H}_{\mathfrak{M}}$ *be the space of Dirac solution in Rindler and Minkowski spacetime respectively. Then there exist*

(i) An isometric embedding $\iota_{\mathfrak{M}} : \mathscr{H}_{\mathfrak{R}} \to, \mathscr{H}_{\mathfrak{M}}$

(ii) A map $\pi_{\mathfrak{R}} : \mathscr{H}_{\mathfrak{M}} \to \mathscr{H}_{\mathfrak{R}}$ *satisfying* $\iota_{\mathfrak{M}} \circ \pi_{\mathfrak{R}} = Id_{\mathscr{H}}$

such that and the orthogonal complement of the image of $\iota_{\mathfrak{M}}$ *coincides with the kernel of* $\pi_{\mathfrak{R}}$.

Proof. In order to extend Dirac solutions from Rindler spacetime to Minkowski space, let $\psi \in \mathscr{H}_{\mathfrak{R}}$ be a solution with spatially compact support. Restricting it to the surface

$$\mathscr{N}_+ := \{(0,x) \in \mathbb{R}^{1,1} \text{ with } x > 0\}$$

gives a smooth function with compact support. We extend this function by zero to the Cauchy surface $\mathscr{N} := \{(0,x) \in \mathbb{R}^{1,1}\}$, namely

$$\Psi_0(x) := \begin{cases} \psi(0,x) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \in C^\infty_0(\mathbb{R}, \mathbb{C}^2)\,.$$

Solving the Cauchy problem in $\mathscr{M}$ with initial data Ψ_0 yields a solution $\Psi(t,x)$ in Minkowski space. We thus obtain an isometric embedding

$$\iota_{\mathscr{M}} : \mathscr{H}_{\mathscr{R}} \to \mathscr{H}_{\mathscr{M}} .$$

It is also useful to introduce the operator $\pi_{\mathscr{R}}$ as the restriction to Rindler spacetime,

$$\pi_{\mathscr{R}} : \mathscr{H}_{\mathscr{M}} \to \mathscr{H}_{\mathscr{R}} , \qquad \pi_{\mathscr{R}} \Psi = \Psi|_{\mathscr{R}} .$$

The identity

$$\pi_{\mathscr{R}} \circ \iota_{\mathscr{M}} = \mathbf{1}_{\mathscr{H}}$$

holds. Moreover, for every $\Psi \in \mathscr{H}_{\mathscr{M}}$ and $\psi \in \mathscr{H}_{\mathscr{R}}$,

$$\big(\Psi \mid \iota_{\mathscr{M}} \psi\big)_{\mathscr{M}} = 2\pi \int_0^\infty \prec \Psi \mid \gamma^0 \psi \succ |_{(0,x)}\, dx = \big(\pi_{\mathscr{R}} \Psi \mid \psi\big)_{\mathscr{R}} ,$$

which can be written as $\iota^*_{\mathscr{M}} = \pi_{\mathscr{R}}$. This relation also shows that the orthogonal complement of the image of $\iota_{\mathscr{M}}$ coincides with the kernel of $\pi_{\mathscr{R}}$, consisting of all Dirac solutions in Minkowski space which vanish on the surface $\mathscr{N}_+$. ∎

3.2 The Relative fermionic Signature Operator

An object of fundamental importance for our purposes is the *spacetime inner product* on Rindler spacetime

$$\langle \cdot \mid \cdot \rangle_{\mathscr{R}} := \int_{\mathscr{R}} \prec \cdot \mid \cdot \succ \, d\mu_g . \tag{3.5}$$

This inner product is not positive definite and, in addition, one should keep in mind that this integral may diverge for solutions of the massive Dirac equation. In analogy to (3.5), the spacetime inner product in Minkowski space is defined by

$$\langle \cdot \mid \cdot \rangle_{\mathscr{M}} := \int_{\mathscr{M}} \prec \cdot \mid \cdot \succ \, d\mu_g .$$

It is not directly related to (3.5) because one integrates over a different spacetime region. However, a direct connection can be obtained by inserting the characteristic function of Rindler spacetime into the integrand,

$$\langle \cdot \mid \cdot \rangle_{\text{Rel}} := \int_{\mathscr{M}} \chi_{\mathscr{R}} \prec \cdot \mid \cdot \succ \, d\mu_g . \tag{3.6}$$

Then for any $\Psi, \tilde{\Psi} \in C^\infty_0(\mathscr{M}, \mathbb{C}^2)$,

$$\big\langle \Psi \mid \tilde{\Psi} \big\rangle_{\text{Rel}} = \big\langle \pi_{\mathscr{R}} \Psi \mid \pi_{\mathscr{R}} \tilde{\Psi} \big\rangle_{\mathscr{M}} .$$

Lemma 3.2.1. *For every $\tilde{\Psi} \in \mathscr{H}_{\mathscr{M}}$, there exists a constant $c = c(\tilde{\Psi})$ such that*

$$\big| \big\langle \Psi \mid \tilde{\Psi} \big\rangle_{\text{Rel}} \big| \le c(\tilde{\Psi})\, \|\Psi\|_{\mathscr{M}} \qquad \forall \Psi \in \mathscr{H}_{\mathscr{M}} . \tag{3.7}$$

Proof. Let $\Phi \in \mathscr{H}_{\mathscr{M}}$. Its restriction to the Cauchy surface $\{t=0\}$ is compact, i.e.

$$\operatorname{supp}(\Phi)(0,.) \subset (-R,R)\,.$$

Due to finite propagation speed (see Corollary 2.3.1) we know that

$$\operatorname{supp}(\Phi)(t,.) \subset (-R-|t|, R+|t|) \qquad \forall t \in \mathbb{R}\,. \tag{3.8}$$

We make use of the fact that solutions of the massive Dirac equation for compactly supported initial data decay rapidly in null directions. More precisely, for any $N \in \mathbb{N}$ there exists a constant $C = C(\Phi, p)$ such that

$$\big|\Phi(t,x)\big| \le \frac{C}{1+|t|^N} \qquad \forall t \in \mathbb{R} \text{ and } x \ge |t|\,. \tag{3.9}$$

This inequality can be verified in two ways. One method is to specialize the more general results in asymptotically flat spacetimes as derived in [Tr15]. Another method is to use that each component of Φ is a solution of the Klein-Gordon equation

$$\big(\partial_t^2 - \partial_x^2 + m^2\big)\Phi(t,x) = 0$$

and to apply the estimates in [Hö97, Theorem 7.2.1], choosing the parameter N in this theorem to be negative and large.

Combining (3.8) and (3.9) with the Schwartz inequality, we obtain the estimate

$$\begin{aligned}\int_{\mathscr{R}} \big| \prec \Psi \mid \tilde{\Psi} \succ \big|\, dt\, dx &\le \int_0^\infty dt \int_{|t|}^{|t|+R} dx\, \|\Psi(t,x)\|\, \|\tilde{\Psi}(t,x)\| \\ &\le \int_0^\infty \|\Psi(t,.)\|_{L^2(dx)}\, \frac{C\sqrt{R}}{1+|t|^N}\, dt = C\,\sqrt{R}\, \|\Psi\| \int_0^\infty \frac{dt}{1+|t|^N}\,.\end{aligned}$$

Choosing $N=2$ gives the desired estimate. ∎

Definition 3.2.1. *Let $\mathscr{D}(S_{\mathrm{Rel}})$ be a subspace of $\mathscr{H}_{\mathscr{M}}$ such that for any $\Psi \in \mathscr{D}(S_{\mathrm{Rel}})$, the anti-linear mapping $\langle \Psi \mid \cdot \rangle_{\mathrm{Rel}} : \mathscr{H}_{\mathscr{M}} \to \mathbb{C}$ is well-defined and bounded, i.e.*

$$\big|\langle \Psi \mid \tilde{\Psi} \rangle_{\mathrm{Rel}}\big| \le c(\Psi)\, \|\tilde{\Psi}\|_{\mathscr{M}}$$

*for a suitable constant $c(\Psi) < \infty$. We call **relative fermionic signature operator** the uniquely densely defined operator $S_{\mathrm{Rel}} : \mathscr{D}(S_{\mathrm{Rel}}) \subset \mathscr{H}_{\mathscr{M}} \to \mathscr{H}_{\mathscr{M}}$ satisfying*

$$\langle \Psi \mid \tilde{\Psi} \rangle_{\mathrm{Rel}} = \big(\Psi \mid S_{\mathrm{Rel}} \tilde{\Psi}\big)_{\mathscr{M}} \qquad \forall \Psi \in \mathscr{H}_{\mathscr{M}}\,. \tag{3.10}$$

Clearly, this definition is well posed, due to the Fréchet-Riesz representation theorem. From (3.10) it is obvious that $S_{\mathscr{R}}$ is symmetric, i.e.

$$\big(\Psi \mid S_{\mathrm{Rel}} \tilde{\Psi}\big)_{\mathscr{M}} = \big(S_{\mathrm{Rel}} \Psi \mid \tilde{\Psi}\big)_{\mathscr{M}}$$

for all $\Psi, \tilde{\Psi} \in \mathscr{H}_{\mathscr{M}}$. We point out that the operator $S_{\mathscr{R}}$ is *unbounded*. This can be understood from the fact that the inequality (3.9) and the subsequent estimate depend essentially on the support of $\tilde{\Psi}$.

In particular, if we consider a sequence of wave functions Ψ_n whose support is shifted more and more to the right,

$$\tilde{\Psi}_n(t,x) = \tilde{\Psi}(t,x-n)\,,$$

then the constant $c(\tilde{\Psi}_n)$ in the statement of Lemma 3.2.1 must be chosen larger and larger if n is increased. This shows that the inequality

$$|\langle \Psi \mid \tilde{\Psi} \rangle_{\text{Rel}}| \leq c\, \|\tilde{\Psi}\|_{\mathscr{M}}\, \|\Psi\|_{\mathscr{M}} \quad \text{for all } \Psi, \tilde{\Psi} \in \mathscr{H}_{\mathscr{M}}$$

is violated, no matter how large the constant c is chosen.

Definition 3.2.2. *Let S_{Rel} be the relative fermionic signature operator and $\iota_{\mathscr{M}}$ and $\pi_{\mathscr{R}}$ as in Lemma 3.1.1. Then we call* ***fermionic signature operator*** *the densely defined operator on $\mathscr{H}_{\mathscr{R}}$ defined by*

$$S = \pi_{\mathscr{R}}\, S_{\text{Rel}}\, \iota_{\mathscr{M}} \qquad \textit{with} \qquad \mathscr{D}(\Sigma) = \pi_{\mathscr{R}}\big(\mathscr{D}(S_{\text{Rel}})\big)\,. \tag{3.11}$$

Our goal is to show that in Rindler spacetime, the domain $\mathscr{D}(S)$ of this operator can be chosen as a dense subset of $\mathscr{H}_{\mathscr{R}}$, and that the fermionic signature operator has a unique self-adjoint extension. Our method is to compute S_{Rel} in more detail in momentum space: As we shall see, the operator S_{Rel} becomes a multiplication operator, making it possible to construct the self-adjoint extension with standard functional analytic methods.

3.2.1 Transformation to Momentum Space

For the following computations, it is most convenient to work in momentum space. We denote the position and momentum variables by $q = (t,x)$ and $p = (\omega,k)$, respectively. Clearly, any smooth and spatially compact Dirac solution $\Psi \in C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M})$ can be represented as

$$\Psi(q) = \int_{\mathbb{R}^2} \frac{d^2p}{(2\pi)^2}\, \hat{\Psi}(p)\, \delta(p^2 - m^2)\, e^{-ipq}\,, \tag{3.12}$$

where $\hat{\Psi}$ is a smooth function on the mass shell (and $pq := \omega t - kx$ is the Minkowski inner product). In this momentum representation, the massive Dirac equation (3.2) reduces to the algebraic equation

$$\big(\gamma^\mu p_\mu - m\big)\hat{\Psi}(p) = 0\,.$$

The matrix $\gamma^\mu p_\mu - m$ has eigenvalues 0 and $-2m$. Its kernel is positive definite with respect to the spin scalar product if p is on the upper mass shell, and it is negative definite if p is on the lower mass shell. Thus we can choose a spinor $\mathfrak{f}(p)$ with the properties

$$(\gamma^\mu p_\mu - m)\,\mathfrak{f}(p) = 0 \qquad \text{and} \qquad \prec \mathfrak{f}(p) | \mathfrak{f}(p) \succ\, = \varepsilon(\omega)\,, \tag{3.13}$$

where ε is the sign function $\varepsilon(\omega) = 1$ for $\omega \geq 0$ and $\varepsilon(\omega) = -1$ otherwise. More specifically, we choose

$$\mathfrak{f}(p) = \frac{1}{\sqrt{2m}}\, \frac{1}{\sqrt{\varepsilon(\omega)(\omega - k)}} \begin{pmatrix} m \\ \omega - k \end{pmatrix}. \tag{3.14}$$

Lemma 3.2.2. *The spinor* $\mathfrak{f}(p)$ *satisfies the relations*

$$\begin{aligned} \prec \mathfrak{f}(\omega,k) \mid \gamma^0 \mathfrak{f}(-\omega,k) \succ &= 0 \\ \prec \mathfrak{f}(\omega,k) \mid \gamma^0 \mathfrak{f}(\omega,k) \succ &= \frac{|\omega|}{m} . \end{aligned}$$

Proof. These relations can be verified in a straightforward manner using the explicit formulas (3.13) and (2.4). Alternatively, they can also be derived abstractly by applying the anti-commutation relations of Dirac matrices:

$$\begin{aligned} \prec \mathfrak{f}(\omega,k) \mid \gamma^0 \mathfrak{f}(-\omega,k) \succ &= \frac{1}{m} \prec \not{p}\, \mathfrak{f}(\omega,k) \mid \gamma^0 \mathfrak{f}(-\omega,k) \succ \\ &= \frac{1}{m} \prec \mathfrak{f}(\omega,k) \mid \big(\omega\gamma^0 - k\gamma^1\big)\gamma^0 \mathfrak{f}(-\omega,k) \succ \\ &= \frac{1}{m} \prec \mathfrak{f}(\omega,k) \mid \gamma^0 \big(\omega\gamma^0 + k\gamma^1\big) \mathfrak{f}(-\omega,k) \succ = - \prec \mathfrak{f}(\omega,k) \mid \gamma^0 \mathfrak{f}(-\omega,k) \succ \\ \prec \mathfrak{f}(\omega,k) \mid \gamma^0 \mathfrak{f}(\omega,k) \succ &= \frac{1}{m} \prec \not{p}\, \mathfrak{f}(\omega,k) \mid \gamma^0 \mathfrak{f}(\omega,k) \succ \\ &= \frac{1}{m} \prec \mathfrak{f}(\omega,k) \mid \gamma^0 \big(\omega\gamma^0 + k\gamma^1\big) \mathfrak{f}(\omega,k) \succ \\ &= \frac{2\omega}{m} \prec \mathfrak{f}(\omega,k) \mid \mathfrak{f}(\omega,k) \succ - \frac{1}{m} \prec \mathfrak{f}(\omega,k) \mid \gamma^0 \not{p}\, \mathfrak{f}(\omega,k) \succ \\ &= \frac{2\omega}{m} \prec \mathfrak{f}(\omega,k) \mid \mathfrak{f}(\omega,k) \succ - \prec \mathfrak{f}(\omega,k) \mid \gamma^0 \mathfrak{f}(\omega,k) \succ . \end{aligned}$$

Using the right relation in (3.13), the result follows. ■

It is convenient to represent the spinor $\hat{\Psi}(p)$ in (3.12) as a complex multiple of the spinor $\mathfrak{f}(p)$. Thus we write the Fourier integral (3.12) as

$$\Psi(q) = \int_{\mathbb{R}^2} \frac{d^2p}{2\pi}\, \varepsilon(\omega)\, \delta(p^2 - m^2)\, g(p)\, \mathfrak{f}(p)\, e^{-ipq} \tag{3.15}$$

with a complex-valued function $g(p)$. In the next two lemmas we specify the regularity of the function $g(p)$ and rewrite the scalar product (3.4) in momentum space.

Lemma 3.2.3. *For every smooth and spatially compact Dirac solution* $\Psi \in \mathscr{H}_{\mathscr{M}}$*, the function* g *in the representation* (3.15) *is a Schwartz function on the mass shells, i.e.*

$$g_\pm(k) := g\big(\pm\sqrt{k^2+m^2}, k\big) \in \mathscr{S}(\mathbb{R}, \mathbb{C}) .$$

Proof. Evaluating (3.15) at $t = 0$ gives

$$\begin{aligned} \Psi(0,x) &= \int_{\mathbb{R}^2} \frac{d^2p}{2\pi}\, \varepsilon(\omega)\, \delta(p^2 - m^2)\, g(p)\, \mathfrak{f}(p)\, e^{ikx} \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sum_\pm \frac{\varepsilon(\omega)}{\sqrt{k^2+m^2}}\, g(p)\, \mathfrak{f}(p)\, e^{ikx} \Big|_{p=\left(\pm\sqrt{k^2+m^2},\,k\right)} \\ i\partial_t \Psi(0,x) &= \int_{\mathbb{R}^2} \frac{d^2p}{2\pi}\, \varepsilon(\omega)\, p_0\, \delta(p^2 - m^2)\, g(p)\, \mathfrak{f}(p)\, e^{ikx} \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sum_\pm g(p)\, \mathfrak{f}(p)\, e^{ikx} \Big|_{p=\left(\pm\sqrt{k^2+m^2},\,k\right)} . \end{aligned}$$

On the other hand, taking the one-dimensional Fourier transform, we know that

$$\Psi(0,x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \hat{\Phi}_0(k)\, e^{ikx} \qquad \text{and} \qquad \partial_t \Psi(0,x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \hat{\Phi}_1(k)\, e^{ikx}$$

for Schwartz functions $\hat{\Phi}_0, \hat{\Phi}_1 \in \mathscr{S}(\mathbb{R}, \mathbb{C}^2)$. Comparing the integrands, we obtain

$$\pm\sqrt{k^2+m^2}\, \hat{\Phi}_0(k) + \hat{\Phi}_1(k) = g_\pm(k)\, \mathfrak{f}\big(\pm\sqrt{k^2+m^2}, k\big)\,.$$

Taking the spin scalar product with $\mathfrak{f}$ and using the right equation in (3.13), we get

$$g_\pm(k) = \prec \Big(\sqrt{k^2+m^2}\, \hat{\Phi}_0(k) \pm \hat{\Phi}_1(k)\Big) \,\Big|\, \mathfrak{f}\big(\pm\sqrt{k^2+m^2}, k\big) \succ .$$

According to (3.14), the spinor $\mathfrak{f}$ is smooth and grows at most linearly for large k (meaning that $\|\mathfrak{f}\|_{\mathbb{C}^2} \le c(1+|k|)$ for a suitable constant c). This gives the result. ∎

Lemma 3.2.4. *In the Fourier representation* (3.15)*, the scalar product* (3.4) *can be written as*

$$(\Psi \,|\, \tilde{\Psi})_{\mathscr{M}} = \frac{1}{2m} \int_{\mathbb{R}^2} \overline{g(p)}\, \tilde{g}(p)\, \delta\big(p^2-m^2\big)\, d^2p\,. \tag{3.16}$$

Proof. We substitute (3.15) into (3.4). In view of the rapid decay of g (see Lemma 3.2.3), we may commute the integrals using Plancherel's theorem to obtain

$$\begin{aligned}
(\Psi \,|\, \tilde{\Psi})_{\mathscr{M}} &= 2\pi \int_{-\infty}^{\infty} dx \int_{\mathbb{R}^2} \frac{d^2p}{2\pi}\, \varepsilon(\omega)\, \delta\big(p^2-m^2\big)\, \overline{g(p)} \\
&\quad \times \int_{\mathbb{R}^2} \frac{d^2\tilde{p}}{2\pi}\, \varepsilon(\tilde{\omega})\, \delta\big(\tilde{p}^2-m^2\big)\, \tilde{g}(\tilde{p}) \prec \mathfrak{f}(p) \,|\, \gamma^0 \mathfrak{f}(\tilde{p}) \succ e^{-i(k-\tilde{k})x} \\
&= 2\pi \int_{\mathbb{R}^2} \frac{d^2p}{2\pi}\, \varepsilon(\omega)\, \delta\big(p^2-m^2\big)\, \overline{g(p)} \int_{\mathbb{R}^2} \frac{d^2\tilde{p}}{2\pi}\, \varepsilon(\tilde{\omega})\, \delta\big(\tilde{p}^2-m^2\big) \\
&\quad \times 2\pi\delta\big(k-\tilde{k}\big)\, \tilde{g}(\tilde{p}) \prec \mathfrak{f}(p) \,|\, \gamma^0 \mathfrak{f}(\tilde{p}) \succ \\
&= \int_{\mathbb{R}^2} d^2p\, \varepsilon(\omega)\, \delta\big(p^2-m^2\big)\, \overline{g(p)} \int_{\mathbb{R}^2} d\tilde{\omega}\, \varepsilon(\tilde{\omega})\, \delta\big(\tilde{\omega}^2-k^2-m^2\big) \\
&\quad \times \tilde{g}\big(\tilde{\omega},k\big) \prec \mathfrak{f}(p) \,|\, \gamma^0 \mathfrak{f}\big(\tilde{\omega},k\big) \succ \\
&= \int_{\mathbb{R}^2} d^2p\, \varepsilon(\omega)\, \delta\big(p^2-m^2\big)\, \overline{g(p)}\, \frac{1}{2|\omega|} \\
&\quad \times \sum_\pm \varepsilon(\pm\omega)\, \tilde{g}\big(\pm\omega,k\big) \prec \mathfrak{f}(p) \,|\, \gamma^0 \mathfrak{f}\big(\pm\omega,k\big) \succ .
\end{aligned}$$

Applying Lemma 3.2.2 gives (3.16). ∎

We choose finally a convenient parametrization of the mass shells.

Proposition 3.2.1. *In the parametrization*

$$\begin{pmatrix} \omega \\ k \end{pmatrix} = ms \begin{pmatrix} \cosh\alpha \\ \sinh\alpha \end{pmatrix} \qquad \text{with} \qquad s \in \{\pm 1\} \text{ and } \alpha \in \mathbb{R}\,, \tag{3.17}$$

the scalar product (3.4) *takes the form*

$$(\Psi \,|\, \tilde{\Psi})_{\mathscr{M}} = \frac{1}{4m} \sum_{s=\pm 1} \int_{-\infty}^{\infty} \overline{g(s,\alpha)}\, \tilde{g}(s,\alpha)\, d\alpha\,. \tag{3.18}$$

The variable α is the rapidity of the momentum of the wave in the rest frame.

Proof of Proposition 3.2.1. We carry out the ω-integration in (3.16),

$$\begin{aligned}\int_{\mathbb{R}^2} \overline{g(p)}\, \tilde{g}(p)\, \delta\big(p^2-m^2\big)\, d^2p &= \sum_{\pm} \int_{-\infty}^{\infty} \frac{dk}{2\sqrt{k^2+m^2}}\, \big(\overline{g}\, \tilde{g}\big)\big|_{\left(\pm\sqrt{k^2+m^2},k\right)} \\ &= \sum_{s=\pm} \int_{-\infty}^{\infty} m \cosh\alpha\, \frac{1}{2m\cosh\alpha}\, \big(\overline{g}\, \tilde{g}\big)\big|_{\left(ms\cosh\alpha,\, ms\sinh\alpha\right)}\, d\alpha \\ &= \frac{1}{2} \sum_{s=\pm} \int_{-\infty}^{\infty} \overline{g(s,\alpha)}\, \tilde{g}(s,\alpha)\, d\alpha\,.\end{aligned}$$

This gives the result. ■

We now compute the relative fermionic signature operator more explicitly in momentum space. The first step is to transform the spacetime inner product to momentum space.

Proposition 3.2.2. *For any $\Psi, \tilde{\Psi} \in \mathscr{H}_{\mathscr{M}}$, the spacetime inner product* (3.6) *takes the form*

$$\langle \Psi \,|\, \tilde{\Psi} \rangle_{\text{Rel}} = \frac{1}{4m} \sum_{s,\tilde{s}=\pm 1} \int_{-\infty}^{\infty} d\alpha\, \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} d\tilde{\alpha}\, I_\varepsilon\big(s,\alpha;\tilde{s},\tilde{\alpha}\big)\, \overline{g(s,\alpha)}\, \tilde{g}(\tilde{s},\tilde{\alpha})\,, \tag{3.19}$$

where I_ε is the kernel

$$I_\varepsilon\big(s,\alpha;\tilde{s},\tilde{\alpha}\big) = \frac{1}{4\pi^2 m} \times \begin{cases} \dfrac{s\cosh\beta}{1-\cosh(2\beta+i\varepsilon s)} & \text{if } s=\tilde{s} \\[2ex] -\dfrac{s\sinh\beta}{1+\cosh(2\beta)} & \text{if } s\neq\tilde{s} \end{cases} \tag{3.20}$$

and

$$\beta := \frac{1}{2}\big(\alpha-\tilde{\alpha}\big)\,. \tag{3.21}$$

Proof. Using the Fourier representation (3.15) in (3.6), we obtain

$$\begin{aligned}\langle \Psi \,|\, \tilde{\Psi} \rangle_{\text{Rel}} &= \int_{\mathscr{M}} dt\, dx\, \chi_{\mathscr{R}}(t,x) \int_{\mathbb{R}^2} \frac{d^2p}{2\pi}\, \varepsilon(\omega)\, \delta(p^2-m^2)\, \overline{g(p)} \\ &\quad \times \int_{\mathbb{R}^2} \frac{d^2\tilde{p}}{2\pi}\, \varepsilon(\tilde{\omega})\, \delta(\tilde{p}^2-m^2)\, \tilde{g}(\tilde{p})\, \prec \mathfrak{f}(p) \,|\, \mathfrak{f}(\tilde{p}) \succ\, e^{i(p-\tilde{p})q} \\ &= \int_{\mathbb{R}^2} \frac{d^2p}{(2\pi)^2}\, \varepsilon(\omega)\, \delta(p^2-m^2)\, \overline{g(p)} \\ &\quad \times \int_{\mathbb{R}^2} d^2\tilde{p}\, \varepsilon(\tilde{\omega})\, \delta(\tilde{p}^2-m^2)\, \tilde{g}(\tilde{p})\, \prec \mathfrak{f}(p) \,|\, \mathfrak{f}(\tilde{p}) \succ\, K(p,\tilde{p})\,,\end{aligned} \tag{3.22}$$

where the kernel $K(p,\tilde{p})$ is defined by

$$K(p,\tilde{p}) = \int_{\mathscr{M}} \chi_{\mathscr{R}}(t,x)\, e^{i(p-\tilde{p})q}\, dt\, dx\,. \tag{3.23}$$

Rewriting the integrals in (3.22) in the parametrization (3.17) (exactly as in the proof of Proposition 3.2.1), we get

$$\langle \Psi \,|\, \tilde{\Psi} \rangle_{\text{Rel}} = \frac{1}{16\pi^2} \sum_{s,\tilde{s}=\pm 1} \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\tilde{\alpha}\, s\tilde{s}\, \overline{g(s,\alpha)}\, \tilde{g}(\tilde{s},\tilde{\alpha})\, \prec \mathfrak{f}(p) \,|\, \mathfrak{f}(\tilde{p}) \succ\, K(p,\tilde{p})\,. \tag{3.24}$$

Applying Lemma 3.2.5 and Lemma 3.2.6 below, the result follows. ■

Comparing (3.18) and (3.19), one can immediately read off the relative fermionic signature operator as defined by (3.10).

Corollary 3.2.1. *For any* $\widetilde{\Psi} \in \mathscr{H}_{\mathscr{M}}$*, the relative fermionic signature operators reads*

$$\big(S_{\text{Rel}}\widetilde{\Psi}\big)(s,\alpha) = \sum_{\tilde{s}=\pm 1} \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} I_\varepsilon\big(s,\alpha;\tilde{s},\tilde{\alpha}\big)\, \tilde{g}(\tilde{s},\tilde{\alpha})\, d\tilde{\alpha}\,.$$

In the following two lemmas we compute the spin scalar product and the kernel in (3.24).

Lemma 3.2.5. *In the parametrization* (3.17)*, the spin scalar product of the spinors* (3.14) *is computed by*

$$\prec \mathfrak{f}(s,\alpha) \mid \mathfrak{f}(\tilde{s},\tilde{\alpha}) \succ = \begin{cases} s\cosh\beta & \text{if } s = \tilde{s} \\ s\sinh\beta & \text{if } s \neq \tilde{s}\,. \end{cases}$$

Proof. Using (3.14) and (3.1), we have

$$\prec \mathfrak{f}(p) \mid \mathfrak{f}(\tilde{p}) \succ = \frac{1}{2}\, \frac{(\omega - k) + (\tilde{\omega} - \tilde{k})}{\sqrt{\varepsilon(\omega)(\omega-k)\,\varepsilon(\tilde{\omega})(\tilde{\omega}-\tilde{k})}}\,.$$

In the parametrization (3.17), we obtain

$$\prec \mathfrak{f}(s,\alpha \mid \mathfrak{f}(\tilde{s},\tilde{\alpha}) \succ = \frac{1}{2}\, \frac{se^{-\alpha} + \tilde{s}e^{-\tilde{\alpha}}}{e^{-\frac{\alpha}{2}-\frac{\tilde{\alpha}}{2}}} = \frac{1}{2}\Big(se^{\beta} + \tilde{s}e^{-\beta}\Big)\,.$$

This gives the result. ■

Lemma 3.2.6. *In the parametrization* (3.17)*, the distribution* $K(p,\tilde{p})$ *defined by* (3.23) *has the form*

$$K(s,\alpha;\tilde{s},\tilde{\alpha}) = \frac{1}{m^2} \times \begin{cases} \displaystyle\lim_{\varepsilon\searrow 0} \frac{1}{1-\cosh(\alpha-\tilde{\alpha}-i\varepsilon s)} & \text{if } s = \tilde{s} \\[2ex] \displaystyle\frac{1}{1+\cosh(\alpha-\tilde{\alpha})} & \text{if } s \neq \tilde{s}\,. \end{cases}$$

Proof. We write first (3.23) as

$$K(p,\tilde{p}) = \int_{\mathbb{R}^2} dt\,dx\, \chi(x-t)\,\chi(x+t)\, e^{i(p-\tilde{p})q}\,. \tag{3.25}$$

Introducing null coordinates

$$u = \frac{1}{2}(t-x) \qquad \text{and} \qquad v = \frac{1}{2}(t+x)$$

as well as the corresponding momenta

$$p_u = \omega - \tilde{\omega} + k - \tilde{k} \qquad \text{and} \qquad p_v = \omega - \tilde{\omega} - k + \tilde{k}\,,$$

we can compute the integrals in (3.25) to obtain

$$\begin{aligned} K(p_u,p_v) &= 2\int_{\mathbb{R}^2} du\,dv\, \chi(-2u)\,\chi(2v)\, e^{i(p_u u + p_v v)} = 2\int_{-\infty}^{0} du\, e^{ip_u u} \int_0^{\infty} dv\, e^{ip_v v} \\ &= 2\lim_{\varepsilon\searrow 0}\int_{-\infty}^{0} du\, e^{ip_u u + \varepsilon u} \lim_{\varepsilon'\searrow 0}\int_0^{\infty} dv\, e^{ip_v v - \varepsilon' v} = 2 \lim_{\varepsilon,\varepsilon'\searrow 0} \frac{1}{p_u - i\varepsilon}\, \frac{1}{p_v + i\varepsilon'}\,. \end{aligned}$$

We express next p_u in the parametrization (3.17),

$$\begin{aligned} p_u &= (\omega + k) - (\tilde{\omega} + \tilde{k}) = ms\big(\cosh(\alpha) + \sinh(\alpha)\big) - m\tilde{s}\big(\cosh(\tilde{\alpha}) + \sinh(\tilde{\alpha})\big) \\ &= m\big(se^{\alpha} - \tilde{s}e^{\tilde{\alpha}}\big) . \end{aligned}$$

This gives

$$\lim_{\varepsilon \searrow 0} \frac{1}{p_u - i\varepsilon} = \lim_{\varepsilon \searrow 0} \frac{1}{m(se^{\alpha} - \tilde{s}e^{\tilde{\alpha}}) - i\varepsilon} . \tag{3.26}$$

We distinguish the two cases $s \neq \tilde{s}$ and $s = \tilde{s}$. In the case $s \neq \tilde{s}$, the denominator in (3.26) is always non-zero. Therefore, we can take the limit $\varepsilon \searrow 0$ pointwise to obtain

$$\lim_{\varepsilon \searrow 0} \frac{1}{p_u - i\varepsilon} = \frac{1}{ms} \frac{1}{e^{\alpha} + e^{\tilde{\alpha}}} = \frac{e^{-\alpha}}{ms} \frac{1}{1 + e^{-2\beta}} \qquad (s \neq \tilde{s}),$$

where β is again given by (3.21). In the remaining case $s = \tilde{s}$, we rewrite (3.26) as

$$\begin{aligned} \lim_{\varepsilon \searrow 0} \frac{1}{p_u - i\varepsilon} &= \frac{1}{ms} \lim_{\varepsilon \searrow 0} \frac{1}{(e^{\alpha} - e^{\tilde{\alpha}}) - i\varepsilon s/m} = \frac{e^{-\alpha}}{ms} \lim_{\varepsilon \searrow 0} \frac{1}{1 - e^{-2\beta} - i\varepsilon s e^{-\alpha}/m} \\ &= \frac{e^{-\alpha}}{ms} \lim_{\delta \searrow 0} \frac{1}{1 - e^{-2\beta} - i\delta s\, e^{-2\beta}} = \frac{e^{-\alpha}}{ms} \lim_{\delta \searrow 0} \frac{1}{1 - e^{-2\beta + i\delta s}} , \end{aligned}$$

where $\delta = \varepsilon e^{-\alpha + 2\beta}/m > 0$. We conclude that

$$\lim_{\varepsilon \searrow 0} \frac{1}{p_u - i\varepsilon} = \frac{e^{-\alpha}}{ms} \lim_{\delta \searrow 0} \frac{1}{1 - e^{-2\beta + i\delta s}} \qquad (s = \tilde{s}) .$$

Treating p_v in the same way, we obtain

$$\lim_{\varepsilon \searrow 0} \frac{1}{p_u - i\varepsilon} = \begin{cases} \dfrac{e^{-\alpha}}{ms} \dfrac{1}{1 + e^{-2\beta}} & \text{if } s \neq \tilde{s} \\ \dfrac{e^{-\alpha}}{ms} \lim\limits_{\varepsilon \searrow 0} \dfrac{1}{1 - e^{-2\beta + i\varepsilon s}} & \text{if } s = \tilde{s} \end{cases} \tag{3.27}$$

$$\lim_{\varepsilon' \searrow 0} \frac{1}{p_v + i\varepsilon'} = \begin{cases} \dfrac{e^{\alpha}}{ms} \dfrac{1}{1 + e^{2\beta}} & \text{if } s \neq \tilde{s} \\ \dfrac{e^{\alpha}}{ms} \lim\limits_{\varepsilon' \searrow 0} \dfrac{1}{1 - e^{2\beta - i\varepsilon' s}} & \text{if } s = \tilde{s}. \end{cases} \tag{3.28}$$

When multiplying (3.27) and (3.28), the fact that both limits $\varepsilon, \varepsilon' \searrow 0$ exist in the distributional sense justifies that we can set $\varepsilon = \varepsilon'$ and then perform the limit. Using (3.21), the result follows. ■

3.2.2 The Self-Adjoint Extension

In Corollary 3.2.1, the relative fermionic signature operator S_{Rel} was represented by an integral operator. Since the kernel $I(s, \alpha; \tilde{s}, \tilde{\alpha})$ only depends on the difference $\alpha - \tilde{\alpha}$ (see (3.20) and (3.21)), we can diagonalize the fermionic operator taking advantage of the map

$$U : \mathscr{H}_{\mathscr{M}} \to L^2(\mathbb{R}, \mathbb{C}^2), \qquad g(s, \alpha) \mapsto \hat{g}(s, \ell) = \frac{1}{\sqrt{8\pi m}} \int_{-\infty}^{\infty} g(s, \alpha)\, e^{i\ell\alpha}\, d\alpha . \tag{3.29}$$

The variable ℓ is the conjugate to the rapidity; to the best of our knowledge it does not have an straightforward physical interpretation. From (3.2.1) and Plancherel's theorem, one sees immediately that this mapping is unitary. Moreover, its inverse is given by

$$U^{-1} : L^2(\mathbb{R},\mathbb{C}^2) \to \mathscr{H}_{\mathscr{M}}\,, \qquad \hat{g}(s,\ell) \mapsto g(s,\alpha) = \sqrt{\frac{2m}{\pi}} \int_{-\infty}^{\infty} \hat{g}(s,\ell)\, e^{-i\ell\alpha}\, d\ell\,.$$

Theorem 3.2.1. *Choosing the domain of definition*

$$\mathscr{D}(S_{\text{Rel}}) = U^{-1}\Big(\Big\{\hat{g} \in L^2(\mathbb{R},\mathbb{C}^2) \quad \text{with} \quad \hat{S}_{\text{Rel}}\,\hat{g} \in L^2(\mathbb{R},\mathbb{C}^2)\Big\}\Big) \tag{3.30}$$

where $(\hat{S}_{\text{Rel}}\hat{g})(\ell) = \hat{S}_{\text{Rel}}(\ell)\,\hat{g}(\ell)$ *is the pointwise multiplication by the matrix*

$$\hat{S}_{\text{Rel}}(\ell) = \frac{\ell}{\pi m} \begin{pmatrix} \dfrac{1}{1+e^{-2\pi\ell}} & -\dfrac{i}{2\cosh(\pi\ell)} \\ \dfrac{i}{2\cosh(\pi\ell)} & \dfrac{1}{1+e^{2\pi\ell}} \end{pmatrix}, \tag{3.31}$$

we obtain the unique self-adjoint extension of the relative fermionic signature operator on $\mathscr{H}_{\mathscr{M}}$. *Its spectrum consists of a pure point spectrum at zero and an absolutely continuous spectrum,*

$$\sigma_{pp}\big(S_{\text{Rel}}\big) = \{0\}\,, \qquad \sigma_{ac}\big(S_{\text{Rel}}\big) = \mathbb{R}\,.$$

It has the spectral decomposition

$$S_{\text{Rel}} = \int_{-\infty}^{\infty} \lambda\, dE_\lambda\,,$$

where the spectral measure dE_λ *is given by*

$$E_I = U^{-1}\Big(\chi_I(0)\hat{K} + \chi_I \hat{L}\Big)U\,.$$

Here $\chi_I(0)$ *and* χ_I *are the characteristic functions for* σ_{pp} *and* σ_{ac} *respectively, while* $\hat{K}$ *and* $\hat{L}$ *are the multiplication operators*

$$\hat{L}(\ell) = \frac{\pi m}{\ell}\,\hat{S}_{\text{Rel}}(\ell) = \begin{pmatrix} \dfrac{1}{1+e^{-2\pi\ell}} & -\dfrac{i}{2\cosh(\pi\ell)} \\ \dfrac{i}{2\cosh(\pi\ell)} & \dfrac{1}{1+e^{2\pi\ell}} \end{pmatrix} \tag{3.32}$$

$$\hat{K}(\ell) = \mathbb{1}_{\mathbb{C}^2} - \hat{L}(\ell) = \begin{pmatrix} \dfrac{e^{-2\pi\ell}}{1+e^{-2\pi\ell}} & \dfrac{i}{2\cosh(\pi\ell)} \\ -\dfrac{i}{2\cosh(\pi\ell)} & \dfrac{e^{2\pi\ell}}{1+e^{2\pi\ell}} \end{pmatrix}. \tag{3.33}$$

Remark 3.2.1. *Notice that the operator* $\hat{S}_{\text{Rel}}$ *is not defined for the value of the mass* $m = 0$. *This implies that we will be able to construct Fock states only for massive Dirac fields.*

We note that the kernel of the operator S_{Rel} as described by the operator $\hat{K}$ consists of all Dirac solutions supported in the region $\mathscr{M} \setminus \mathscr{R}$ outside the Rindler wedge. This will be explained in details in the proof of Theorem 3.3.1 below. The proof of this lemma will be given later in this section. Before, we need to infer a relation concerning the integral kernel I_ε.

Lemma 3.2.7. *The integral kernel I_ε, (3.20), satisfies the relation*

$$\lim_{\varepsilon\searrow 0}\int_{-\infty}^{\infty} I_\varepsilon\big(s,\alpha;\tilde{s},\tilde{\alpha}\big)\, e^{-i\ell\tilde{\alpha}}\, d\tilde{\alpha} = e^{-i\ell\alpha}\,\frac{\ell}{2\pi m}\times\begin{cases} \dfrac{2}{1+e^{-2\pi s\ell}} & \text{if } s=\tilde{s} \\ -\dfrac{is}{\cosh(\pi\ell)} & \text{if } s\neq\tilde{s}\,. \end{cases}$$

Proof. In the case $s\neq\tilde{s}$, we have

$$\begin{aligned}&\lim_{\varepsilon\searrow 0}\int_{-\infty}^{\infty} I_\varepsilon\big(s,\alpha;\tilde{s},\tilde{\alpha}\big)\, e^{-i\ell(\tilde{\alpha}-\alpha)}\, d\tilde{\alpha} \\ &\quad= 2\lim_{\varepsilon\searrow 0}\int_{-\infty}^{\infty} I_\varepsilon\big(s,\alpha;\tilde{s},\tilde{\alpha}\big)\, e^{2i\ell\beta}\, d\beta = -\frac{s}{2\pi^2 m}\int_{-\infty}^{\infty}\frac{\sinh\beta}{1+\cosh(2\beta)}\, e^{2i\ell\beta}\, d\beta\,.\end{aligned}$$

The integral can be computed as follows. First, using the transformation

$$\frac{\sinh\beta}{1+\cosh(2\beta)} = -\frac{d}{d\beta}\Big(\frac{1}{e^\beta+e^{-\beta}}\Big)\,,$$

we can integrate by parts to obtain

$$\int_{-\infty}^{\infty}\frac{\sinh\beta}{1+\cosh(2\beta)}\, e^{2i\ell\beta}\, d\beta = 2i\ell\int_{-\infty}^{\infty}\frac{e^{2i\ell\beta}}{e^\beta+e^{-\beta}}\, d\beta = i\ell\int_{-\infty}^{\infty}\frac{e^{2i\ell\beta}}{\cosh\beta}\, d\beta\,.$$

The last integral can be calculated with residues, noticing, in addition that is odd under β-reflection. Therefore, it suffices to consider the case $\ell>0$. Then we can close the contour in the upper half plane. There the integrand has poles at $\beta_n = i\pi(n+\frac{1}{2})$ with $n\in\mathbb{N}_0$. This gives

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{\sinh\beta}{1+\cosh(2\beta)}\, e^{2i\ell\beta}\, d\beta &= -2\pi\ell\sum_{n=0}^{\infty}\text{Res}\Big(\frac{e^{2i\ell\beta}}{\cosh\beta},\beta_n\Big) \\ &= -2\pi\ell\sum_{n=0}^{\infty}(-i)(-1)^n\, e^{-2\pi\ell(n+\frac{1}{2})} = 2\pi i\ell\, e^{-\pi\ell}\sum_{n=0}^{\infty}\big(-e^{-2\pi\ell}\big)^n \\ &= 2\pi i\ell\, e^{-\pi\ell}\,\frac{1}{1+e^{-2\pi\ell}} = \frac{i\pi\ell}{\cosh(\pi\ell)}\,.\end{aligned}\tag{3.34}$$

In the case $s=\tilde{s}$, we find similarly

$$\lim_{\varepsilon\searrow 0}\int_{-\infty}^{\infty} I_\varepsilon\big(s,\alpha;\tilde{s},\tilde{\alpha}\big)\, e^{-i\ell(\tilde{\alpha}-\alpha)}\, d\tilde{\alpha} = \frac{s}{2\pi^2 m}\lim_{\varepsilon\searrow 0}\int_{-\infty}^{\infty}\frac{\cosh(\beta-\frac{i\varepsilon s}{2})}{1-\cosh(2\beta-i\varepsilon s)}\, e^{2i\ell\beta}\, d\beta\,.$$

Rewriting the integrand as

$$\frac{\cosh(\beta-\frac{i\varepsilon s}{2})}{1-\cosh(2\beta-i\varepsilon s)} = -\frac{e^{\beta-\frac{i\varepsilon s}{2}}+e^{-\beta+\frac{i\varepsilon s}{2}}}{(e^{\beta-\frac{i\varepsilon s}{2}}-e^{-\beta+\frac{i\varepsilon s}{2}})^2} = \frac{d}{d\beta}\Big(\frac{1}{e^{\beta-\frac{i\varepsilon s}{2}}-e^{-\beta+\frac{i\varepsilon s}{2}}}\Big)\,,$$

we can integrate again by parts to obtain

$$\int_{-\infty}^{\infty}\frac{\cosh(\beta-\frac{i\varepsilon s}{2})}{1-\cosh(2\beta-i\varepsilon s)}\, e^{2i\ell\beta}\, d\beta = -i\ell\int_{-\infty}^{\infty}\frac{e^{2i\ell\beta}}{\sinh(\beta-\frac{i\varepsilon s}{2})}\, d\beta\,.\tag{3.35}$$

The last integral is odd under the joint transformations

$$\ell\mapsto -\ell \qquad\text{and}\qquad s\mapsto -s\,.$$

Therefore, it suffices again to consider the case $\ell > 0$, where the contour can be closed in the upper half plane. In the case $s = 1$, the contour encloses the poles at the points $\beta_n = i\pi n$ with $n \in \mathbb{N}_0$. This gives

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\int_{-\infty}^{\infty} \frac{\cosh\beta}{1-\cosh(2\beta - i\varepsilon)}\, e^{2i\ell\beta}\, d\beta = 2\pi\ell \sum_{n=0}^{\infty} \text{Res}\Big(\frac{e^{2i\ell\beta}}{\sinh\beta}, \beta_n\Big) \\
&= 2\pi\ell \sum_{n=0}^{\infty} \big(-e^{-2\pi\ell}\big)^n = \frac{2\pi\ell}{1+e^{-2\pi\ell}}.
\end{aligned}$$

In the case $s = -1$, the contour does not enclose the pole at $\beta_0 = 0$. We thus obtain

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\int_{-\infty}^{\infty} \frac{\cosh\beta}{1-\cosh(2\beta - i\varepsilon)}\, e^{2i\ell\beta}\, d\beta = 2\pi\ell \sum_{n=1}^{\infty} \text{Res}\Big(\frac{e^{2i\ell\beta}}{\sinh\beta}, \beta_n\Big) \\
&= 2\pi\ell \sum_{n=1}^{\infty} \big(-e^{-2\pi\ell}\big)^n = -2\pi\ell\frac{e^{-2\pi\ell}}{1+e^{-2\pi\ell}} = -\frac{2\pi\ell}{1+e^{2\pi\ell}}.
\end{aligned}$$

This concludes the proof. ■

Proof of Theorem 3.2.1. For a Dirac solution $\Psi \in \mathscr{H}_{\mathscr{M}}$, we know from Lemma 3.2.3 and Proposition 3.2.1 that the corresponding function $g(s,\alpha)$ is smooth and that all its derivatives are square integrable. As a consequence, its Fourier transform is pointwise bounded and has rapid decay, i.e.

$$\sup_{\ell} \big|(1+\ell^2)^p\, \hat{g}(\ell)\big| < \infty \qquad \text{for all } p \in \mathbb{N}.$$

Using furthermore that the kernel $I_\varepsilon(s,\alpha,\tilde{s},\cdot)$ given in (3.20) decays exponentially, we may use Fubini's theorem to exchange the orders of integration in the following computation,

$$\begin{aligned}
\big(S_{\text{Rel}}\Psi\big)(s,\alpha) &= \sum_{\tilde{s}=\pm 1} \lim_{\varepsilon\searrow 0}\int_{-\infty}^{\infty} I_\varepsilon\big(s,\alpha;\tilde{s},\tilde{\alpha}\big)\sqrt{\frac{2m}{\pi}}\Big(\int_{-\infty}^{\infty}\hat{g}(s,\ell)\, e^{-i\ell\tilde{\alpha}}\, d\ell\Big)\, d\tilde{\alpha} \\
&= \sqrt{\frac{2m}{\pi}} \sum_{\tilde{s}=\pm 1}\int_{-\infty}^{\infty}\hat{g}(s,\ell)\Big(\lim_{\varepsilon\searrow 0}\int_{-\infty}^{\infty} I_\varepsilon\big(s,\alpha;\tilde{s},\tilde{\alpha}\big)\, e^{-i\ell\tilde{\alpha}}\, d\alpha\Big) d\ell \\
&= \sqrt{\frac{2m}{\pi}} \sum_{\tilde{s}=\pm 1}\int_{-\infty}^{\infty}\big(\hat{S}_{\text{Rel}}(\ell)\,\hat{g}(\ell)\big)_s\, e^{-i\ell\alpha}\, d\ell = \big(U^{-1}\hat{S}_{\text{Rel}}\, U\Psi\big)(s,\alpha),
\end{aligned}$$

where in the last line we applied Lemma 3.2.7. Therefore, the unitary transformation of S_{Rel} yields a multiplication operator, i.e.

$$\big(U\, S_{\text{Rel}}\, U^{-1}\hat{g}\big)(\ell) = \hat{S}_{\text{Rel}}(\ell)\,\hat{g}(\ell) \qquad \text{for all } \hat{g} \in U\big(C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M}) \cap \mathscr{H}\big).$$

Such operator can be extended to the domain

$$\mathscr{D}\big(\hat{S}_{\text{Rel}}\big) := \Big\{\hat{g} \in L^2(\mathbb{R},\mathbb{C}^2) \quad \text{with} \quad \hat{S}_{\text{Rel}}\,\hat{g} \in L^2(\mathbb{R},\mathbb{C}^2)\Big\} \tag{3.36}$$

where again $(\hat{S}_{\text{Rel}}\hat{g})(l) := \hat{S}_{\text{Rel}}(\ell)\,\hat{g}(\ell)$. Notice that the matrix

$$\hat{S}_{\text{Rel}}(\ell) = \frac{\ell}{\pi m}\begin{pmatrix} \dfrac{1}{1+e^{-2\pi\ell}} & -\dfrac{i}{2\cosh(\pi\ell)} \\ \dfrac{i}{2\cosh(\pi\ell)} & \dfrac{1}{1+e^{2\pi\ell}} \end{pmatrix} \tag{3.37}$$

has the eigenvalues

$$\lambda = 0 \qquad \text{and} \qquad \lambda = \ell/\pi m \tag{3.38}$$

with respective eigenfunctions

$$\hat{g}(\ell) = \begin{pmatrix} ie^{-\pi\ell} \\ 1 \end{pmatrix} \qquad \text{and} \qquad \hat{g}(\ell) = \begin{pmatrix} -ie^{\pi\ell} \\ 1 \end{pmatrix} . \tag{3.39}$$

Our task is to prove that with this domain, the multiplication operator $\hat{S}_{\text{Rel}}$ is self-adjoint. Once this has been shown, we obtain the self-adjointness of S_{Rel} with domain (3.30) by a unitary transformation. Moreover, the properties of the spectrum and the spectral measure follow immediately by computing the spectral measure of the multiplication operator $\hat{S}_{\text{Rel}}$ and unitarily transforming back to the Hilbert space $\mathscr{H}$.

In order to establish that the multiplication operator $\hat{S}_{\text{Rel}}$ with domain (3.36) is self-adjoint, we need to show that the domain of its adjoint $\hat{S}^*_{\text{Rel}}$ coincides with (3.36). This follows using standard functional methods (see for example [La02, Mo13]), which we here recall for completeness: $\forall \Psi \in \mathscr{D}(\hat{S}^*_{\text{Rel}})$ it holds

$$\langle \Psi, \hat{S}_{\text{Rel}} u \rangle_{L^2(\mathbb{R},\mathbb{C}^2)} = \langle \hat{S}^*_{\text{Rel}} \Psi, u \rangle_{L^2(\mathbb{R},\mathbb{C}^2)} \qquad \text{for all } u \in \mathscr{D}\big(\hat{S}_{\text{Rel}}\big) .$$

Since the function $\hat{S}^*_{\text{Rel}}\Psi$ is in $L^2(\mathbb{R},\mathbb{C}^2)$, we may apply Lebesgue's monotone convergence theorem to obtain

$$\begin{aligned}
\big\|\hat{S}^*_{\text{Rel}}\Psi\big\|_{L^2(\mathbb{R},\mathbb{C}^2)} &= \lim_{L\to\infty} \big\|\chi_{[-L,L]}\,\hat{S}^*_{\text{Rel}}\Psi\big\|_{L^2(\mathbb{R},\mathbb{C}^2)} \\
&= \lim_{L\to\infty} \sup_{\Phi\in\mathscr{H},\,\|\Phi\|=1} \big\langle \Phi, \chi_{[-L,L]}\,\hat{S}^*_{\text{Rel}}\Psi\big\rangle_{L^2(\mathbb{R},\mathbb{C}^2)} \\
&\overset{(*)}{=} \lim_{L\to\infty} \sup_{\Phi\in\mathscr{H},\,\|\Phi\|=1} \big\langle \hat{S}_{\text{Rel}}\,\chi_{[-L,L]}\,\Phi, \Psi\big\rangle_{L^2(\mathbb{R},\mathbb{C}^2)} \\
&= \lim_{L\to\infty} \sup_{\Phi\in\mathscr{H},\,\|\Phi\|=1} \int_{-L}^{L} \big\langle \Phi(\ell), \hat{S}_{\text{Rel}}(\ell)\,\Psi(\ell)\big\rangle_{\mathbb{C}^2}\, d\ell \\
&= \lim_{L\to\infty} \bigg(\int_{-L}^{L} \big\|\hat{S}_{\text{Rel}}(\ell)\,\Psi(\ell)\big\|^2_{\mathbb{C}^2}\, d\ell \bigg)^{\frac{1}{2}} ,
\end{aligned}$$

where in $(*)$ we used that the function $\chi_{[-L,L]}\Phi$ lies in the domain of $\hat{S}_{\text{Rel}}$ (see (3.36) and exploit the fact that the matrix $\hat{S}_{\text{Rel}}(\ell)$ in (3.37) is uniformly bounded for $\ell \in [-L,L]$). Applying again Lebesgue's monotone convergence theorem, we infer that the pointwise product $\hat{S}_{\text{Rel}}(\ell)\,\Psi(\ell)$ is in $L^2(\mathbb{R},\mathbb{C}^2)$. Using (3.36), it follows that the vector Ψ lies in the domain of $\hat{S}_{\text{Rel}}$. This concludes the proof. ■

3.3 The Fermionic Signature Operator of Rindler Spacetime

Having defined the relative fermionic signature operator S_{Rel} as a self-adjoint operator with dense domain $\mathscr{D}(S_{\text{Rel}})$, the fermionic signature operator S in Rindler spacetime is obtained from (3.11). We then have the following result.

Theorem 3.3.1. *Choosing the domain of definition*

$$\mathscr{D}(S) = \pi_{\mathscr{R}}\, \mathscr{D}(S_{\mathrm{Rel}}) \tag{3.40}$$

(with $\mathscr{D}(S_{\mathrm{Rel}})$ *according to* (3.30)*), the fermionic signature operator* S *in Rindler spacetime is a self-adjoint operator on* $\mathscr{H}_{\mathscr{R}}$*. It has an absolutely continuous spectrum with spectral measure* dE_λ *given by*

$$E_I = \pi_{\mathscr{R}}\, U^{-1}\big(\chi_I \widehat{L}\big) U\, \iota_{\mathscr{M}}\,,$$

where $\widehat{L}$ *is again the multiplication operator* (3.33).

Proof. On the solution space $\mathscr{H}_{\mathscr{M}}$ in Minkowski space, we consider the transformation

$$T_{\mathrm{CPT}} : \mathscr{H}_{\mathscr{M}} \to \mathscr{H}_{\mathscr{M}}\,, \qquad \Psi(t,x) \mapsto \gamma^0 \gamma^1\, \Psi(-t,-x)$$

(in physics referred to as the CPT transformation [BD64, Section 5.4]; one verifies directly that this transformation maps again to solutions of the massive Dirac equation). A direct computation shows that T_{CPT} is unitary and that $T_{\mathrm{CPT}}^2 = -\mathbf{1}$.

The transformation T_{CPT} can be used to describe the Hilbert space $\mathscr{H}_{\mathscr{M}}$ completely in terms of $\mathscr{H}_{\mathscr{R}}$. To see how this comes about, we first note that a solution $\Psi \in \mathscr{H}_{\mathscr{M}}$ is determined uniquely by its Cauchy data at time zero. The restriction to the right half line $\Psi|_{\{t=0,x>0\}}$ gives rise to a unique solution in $\mathscr{H}_{\mathscr{R}}$, and applying $\iota_{\mathscr{M}}$ yields a solution in Minkowski space which vanishes identically on the left half line $\Psi|_{\{t=0,x<0\}}$. Applying T_{CPT} to this solution gives a new solution which vanishes identically on the right half line $\Psi|_{\{t=0,x>0\}}$. In view of (3.4), the solutions which vanish on the right half line are orthogonal to those which vanish on the left half line. We thus obtain the orthogonal direct sum decomposition

$$\mathscr{H}_{\mathscr{M}} = \big(T_{\mathrm{CPT}}\, \iota_{\mathscr{M}}\, \mathscr{H}_{\mathscr{R}}\big) \oplus \big(\iota_{\mathscr{M}}\, \mathscr{H}_{\mathscr{R}}\big)\,.$$

Since the Dirac solutions in $T_{\mathrm{CPT}}\, \iota_{\mathscr{M}}\, \mathscr{H}_{\mathscr{R}}$ vanish identically in the Rindler wedge, it is obvious that

$$S_{\mathrm{Rel}}\big|_{T_{\mathrm{CPT}}\, \iota_{\mathscr{M}}\, \mathscr{H}_{\mathscr{R}}} = 0 \qquad \text{and} \qquad S_{\mathrm{Rel}}\big(\mathscr{H}_{\mathscr{M}}\big) \subset \iota_{\mathscr{M}}\, \mathscr{H}_{\mathscr{R}}\,.$$

Moreover, writing T_{CPT} in momentum space, one sees that it leaves the parameter ℓ in (3.29) unchanged, mapping the trivial and non-trivial eigenspaces of the matrix (3.37) to each other (see (3.38) and (3.39)). This shows that the operator $\iota_{\mathscr{M}}$ in (3.11) maps precisely to the orthogonal complement of the kernel of S_{Rel}, and that the image of S_{Rel} is mapped by $\pi_{\mathscr{M}}$ unitarily to $\mathscr{H}_{\mathscr{R}}$. Therefore, the spectral representation of S is obtained by that of S_{Rel} simply by removing the kernel. This gives the result. ∎

3.3.1 Connection to the Hamiltonian in Rindler Coordinates

The fermionic signature operator is closely related to the Dirac Hamiltonian in Rindler coordinates, as we now explain. We already saw in Example 2.1, that in the Rindler coordinates $\tau \in \mathbb{R}$ and $\rho \in (0,\infty)$, translations in the time coordinate τ,

$$\tau \mapsto \tau + \Delta\,, \qquad \rho \mapsto \rho\,, \tag{3.41}$$

describe a Killing symmetry. Therefore, writing the massive Dirac equation in this time coordinate in the Hamiltonian form

$$i\partial_\tau \psi = H\psi\,, \tag{3.42}$$

the Dirac Hamiltonian is time independent (for details see the proof of Theorem 3.3.2 below).

Theorem 3.3.2. *The fermionic signature operator S and the Hamiltonian H in Rindler coordinates satisfy the relation*

$$S = -\frac{H}{\pi m}\,.$$

Proof. One method of deriving the Dirac operator would be to compute the spin connection in this coordinate system. For our purposes, it is more convenient to take again the Dirac operator in the reference frame (t,x) and to express it in the Rindler coordinates (τ,ρ), without transforming the spinor basis (this Dirac operator coincides with the intrinsic Dirac operator up to a local U(1,1)-gauge transformation; for details in the more general four-dimensional setting see [Fi98]). Using the identities

$$\begin{aligned}
\frac{\partial}{\partial\rho} &= \frac{\partial t}{\partial\rho}\frac{\partial}{\partial t} + \frac{\partial x}{\partial\rho}\frac{\partial}{\partial x} = \sinh\tau\,\frac{\partial}{\partial t} + \cosh\tau\,\frac{\partial}{\partial x} \\
\frac{\partial}{\partial\tau} &= \frac{\partial t}{\partial\tau}\frac{\partial}{\partial t} + \frac{\partial x}{\partial\tau}\frac{\partial}{\partial x} = \rho\cosh\tau\,\frac{\partial}{\partial t} + \rho\sinh\tau\,\frac{\partial}{\partial x}\,,
\end{aligned}$$

the Dirac operator becomes

$$\begin{aligned}
\mathscr{D} &= \frac{i}{\rho}\Big(\gamma^0\cosh\tau - \gamma^1\sinh\tau\Big)\partial_\tau + i\Big(-\gamma^0\sinh\tau + \gamma^1\cosh\tau\Big)\partial_\rho \\
&= \frac{i}{\rho}\begin{pmatrix} 0 & e^{-\tau} \\ e^{\tau} & 0 \end{pmatrix}\partial_\tau + i\begin{pmatrix} 0 & e^{-\tau} \\ -e^{\tau} & 0 \end{pmatrix}\partial_\rho\,.
\end{aligned}$$

Consequently, the Dirac Hamiltonian in (3.42) can be written as

$$H = i\rho\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\partial_\rho + m\rho\begin{pmatrix} 0 & e^{-\tau} \\ e^{\tau} & 0 \end{pmatrix}.$$

The time translation in (3.41) must be complemented by the corresponding transformation of the spinors

$$\psi \mapsto \exp\Big(\gamma^0\gamma^1\,\frac{\Delta}{2}\Big)\psi = \begin{pmatrix} e^{-\frac{\Delta}{2}} & 0 \\ 0 & e^{\frac{\Delta}{2}} \end{pmatrix}\psi\,. \tag{3.43}$$

Indeed, by direct computation one verifies that the Dirac operator as well as the Dirac Hamiltonian are invariant under the joint transformations (3.41) and (3.43). If we also change the momentum variables according to

$$\alpha \mapsto \alpha + \Delta\,, \tag{3.44}$$

we know by Lorentz symmetry that the Dirac solutions in our Fourier representation remain unchanged. Therefore, the time evolution in the time coordinate τ is described by the inverse of the transformation (3.44), $\alpha \mapsto \alpha - \Delta$. We conclude that, infinitesimally, the Hamiltonian H is given by $i\partial_\tau = i\partial_\Delta = -i\partial_\alpha$. Using this formula in the plane wave ansatz

$$g(s,\alpha) = e^{-i\ell\alpha} g(s,\ell),$$

we conclude that

$$H\widehat{g}(s,\ell) = -\ell\,\widehat{g}(s,\ell).$$

Comparing with (3.38), one sees that the eigenvalues of H agree up to a factor $-\pi m$ with those of the relative fermionic signature operator. Taking into account that the image of the operator $\iota_{\mathscr{M}}$ in (3.11) coincides with the orthogonal complement of the kernel of S_{Rel} (see Theorem 3.3.1), we obtain the result. ■

3.4 The FP States and Thermal States

As we explained in [FL15, FMR16a], the fermionic signature operator can also be used to single out a distinguished fermionic quantum state, sometimes referred to as the *fermionic projector state*, for short *FP state*. We now recall the construction and show that, in the two-dimensional Rindler spacetime, this construction gives precisely the Fulling-Rindler vacuum. We work again intrinsically in Rindler spacetime. Since the Hamiltonian in the massive Dirac equation (3.42) is independent of τ, we can separate the τ-dependence with a plane wave ansatz

$$\psi(\tau,\rho) = e^{-i\Omega\tau}\chi(\rho).$$

The sign of the separation constant Ω gives a splitting of the solution space of the massive Dirac equation into two subspaces. The Fulling-Rindler vacuum is the unique quantum state corresponding to this "frequency splitting" in the time coordinate τ. Next, the fermionic signature operator as defined by (3.10) is a self-adjoint operator with dense domain $\mathscr{D}(S)$ given by (3.40). Therefore, the functional calculus gives rise to projection operators

$$\Pi_S^- := \chi_{(-\infty,0)}(S) = \int_{-\infty}^0 \lambda\,dE_\lambda \quad \text{and} \quad \Pi_S^+ := \chi_{(0,\infty)}(S) = \int_0^\infty \lambda\,dE_\lambda,$$

where dE_λ is an absolutely continuous spectrum with spectral measure given by

$$E_U = \pi_{\mathscr{R}}\,U^{-1}\big(\chi_U\widehat{L}\big)U\,\iota_{\mathscr{M}},$$

and $\widehat{L}$ is again the multiplication operator (3.33). Applying Corollary 2.4.2 and Lemma 2.4.1 we obtain the FP state ω_{Π_S}, namely a pure quasifree state on the algebra of Dirac fields $\mathfrak{F}(\mathscr{R})$ and, as a consequence, on $\mathfrak{F}_{\text{osb}}(\mathscr{R})$ - see Definition 2.4.7 and 2.4.8. In view of Theorem 3.3.2, the projection operators Π_S^- and Π_S^+ coincide with the above frequency splitting. We thus obtain the following result:

Corollary 3.4.1. *The FP states $\omega_{\Pi_S^-}$ and $\omega_{\Pi_S^+}$ coincides with the Fulling-Rindler vacuum. Moreover, the quasifree state ω_W associated to the positive operator*

$$W_\beta = \frac{1}{1+e^{\beta m\pi S}},$$

is a KMS state of inverse temperature β. Choosing $\beta = 2\pi$, we get the Unruh state.

Proof. Using Theorem 3.3.2, Corollary 2.4.2 and Theorem 2.4.4, we can conclude. ■

3.5 Extension to Four-Dimensional Rindler Spacetime

We now explain how our results extend to the case of four-dimensional Rindler spacetime. Let $\mathscr{M} = \mathbb{R}^{1,3}$ be the four-dimensional Minkowski space and let be $\mathscr{R}$ the subset

$$\mathscr{R} = \left\{(t,x,y,z) \in \mathbb{R}^{1,3} \text{ with } |t| < x\right\}.$$

The massive Dirac equation in Rindler spacetime is formulated as the restriction of the massive Dirac equation in Minkowski space to $\mathscr{R}$ (we use the same notation and conventions as in [BD64]). Its solutions are most easily constructed by separating the y-and z-dependence with a plane wave ansatz,

$$\psi(t,x,y,z) = e^{ik_y y + ik_z z}\, \tilde{\psi}(t,x)\,, \tag{3.45}$$

giving the massive Dirac equation in t and x

$$\left(i\gamma^0\partial_t + i\gamma^1\partial_x\right)\tilde{\psi}(t,x) = \left(m + \gamma^2 k_y + \gamma^3 k_z\right)\tilde{\psi}(t,x)\,.$$

Transforming to momentum space, the solutions lie on the mass shell

$$\tilde{m} := \sqrt{m^2 + k_y^2 + k_z^2}\,. \tag{3.46}$$

Similarly to (3.12), we can make the ansatz

$$\tilde{\psi}(q) = \int_{\mathbb{R}^2} \frac{d^2p}{(2\pi)^2}\, \hat{\psi}(p)\, \delta(p^2 - \tilde{m}^2)\, e^{-ipq}\,,$$

giving rise to the algebraic equation

$$\left(\omega\gamma^0 - k\gamma^1\right)\hat{\psi} = \left(m + \gamma^2 k_y + \gamma^3 k_z\right)\hat{\psi} \tag{3.47}$$

(where again $p = (\omega, k)$). This equation has a two-dimensional solution space. In analogy to (3.13), we choose a basis of solutions $\mathfrak{f}_1, \mathfrak{f}_2$. In the next lemma it is shown that these spinors can be chosen to have similar properties to those stated in Lemma 3.2.2 and Lemma 3.2.5.

Lemma 3.5.1. *Given k_y and k_z, there are spinors $\mathfrak{f}_a(p)$ with $a = \pm 1$ which solve the equation* (3.47) *and satisfy the relations*

$$\begin{aligned} \prec \mathfrak{f}_a(\omega,k) | \mathfrak{f}_b(\omega,k) \succ &= \varepsilon(\omega)\, \delta_{ab} \\ \prec \mathfrak{f}_a(\omega,k) \mid \gamma^0\, \mathfrak{f}_b(-\omega,k) \succ &= 0 \\ \prec \mathfrak{f}_a(\omega,k) \mid \gamma^0\, \mathfrak{f}_b(\omega,k) \succ &= \frac{|\omega|}{m}\, \delta_{ab}\,. \end{aligned}$$

Moreover, in the parametrization (3.17)*,*

$$\prec \mathfrak{f}_a(s,\alpha) | \mathfrak{f}_b(\tilde{s},\tilde{\alpha}) \succ = s\, \delta_{ab}\, \frac{\tilde{m}}{m} \begin{cases} \cosh(\beta + i\nu_a) & \text{if } s = \tilde{s} \\ \sinh(\beta + i\nu_a) & \text{if } s \neq \tilde{s}\,, \end{cases}$$

where β is again given by (3.21)*, and the angle $\nu_a \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is defined by*

$$\nu_a = \arctan\left(\frac{a}{m}\sqrt{k_y^2 + k_z^2}\right). \tag{3.48}$$

Proof. After rotating the reference frame, we can assume that $k_z = 0$ and $k_y > 0$. Then, in the Dirac representation (see for example [BD64]), the massive Dirac equation (3.47) takes the form

$$\begin{pmatrix} \omega - m & 0 & 0 & -k + ik_y \\ 0 & \omega - m & -k - ik_y & 0 \\ 0 & k - ik_y & -\omega - m & 0 \\ k + ik_y & 0 & 0 & -\omega - m \end{pmatrix} \hat{\psi} = 0 . \tag{3.49}$$

This matrix has two invariant subspaces: one spanned by the first and fourth spinor components, and the other spanned by the second and third spinor components. Choosing $\mathfrak{f}_1$ in the first and $\mathfrak{f}_{-1}$ in the second of these subspaces, the above inner products all vanish if $a \neq b$. In the remaining case $a = b$, one can restrict the attention to two-spinors. In order to get back to the framework in two-dimensional Rindler spacetime, we use the identity

$$U \begin{pmatrix} \omega - m & -k \pm ik_y \\ k \pm ik_y & -\omega - m \end{pmatrix} U = \begin{pmatrix} \omega - \tilde{m} & -k \\ k & -\omega - \tilde{m} \end{pmatrix} ,$$

where U is the matrix

$$U = \begin{pmatrix} \cos(\nu_a/2) & i\sin(\nu_a/2) \\ i\sin(\nu_a/2) & \cos(\nu_a/2) \end{pmatrix} .$$

Now the results follow by direct computation. ■

Using the result of this lemma, we can represent the solution in analogy to (3.15) by

$$\psi(q) = \sum_{a=\pm 1} \int_{\mathbb{R}^2} \frac{d^2p}{2\pi}\, \varepsilon(\omega)\, \delta(p^2 - m^2)\, g_a(p)\, \mathfrak{f}_a(p)\, e^{-ipq}$$

with two complex-valued functions $g_{\pm 1}$. The subsequent analysis can be extended in a straightforward way. In particular, the kernel I_ε in Corollary 3.2.1 ought to be replaced by the kernels

$$I^a_\varepsilon\big(s,\alpha;\tilde{s},\tilde{\alpha}\big) = \frac{1}{4\pi^2 m} \times \begin{cases} \dfrac{s\cosh(\beta + i\nu_a)}{1 - \cosh(2\beta + i\varepsilon s)} & \text{if } s = \tilde{s} \\[2ex] -\dfrac{s\sinh(\beta + i\nu_a)}{1 + \cosh(2\beta)} & \text{if } s \neq \tilde{s} , \end{cases}$$

where again $2\beta = \alpha - \tilde{\alpha}$. The residues can be computed as in Lemma 3.2.7 if one transforms the integrals in the following way,

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{\sinh(\beta + i\nu_a)}{1 + \cosh(2\beta)}\, e^{-2i\ell\beta}\, d\beta \\ &= \cos\nu_a \int_{-\infty}^{\infty} \frac{\sinh(\beta)}{1 + \cosh(2\beta)}\, e^{-2i\ell\beta}\, d\beta + i\sin\nu_a \int_{-\infty}^{\infty} \frac{\cosh(\beta)}{1 + \cosh(2\beta)}\, e^{-2i\ell\beta}\, d\beta \\ &= -\cos\nu_a \int_{-\infty}^{\infty} \frac{d}{d\beta} \left(\frac{1}{e^\beta + e^{-\beta}} \right) e^{-2i\ell\beta}\, d\beta + i\sin\nu_a \int_{-\infty}^{\infty} \frac{1}{e^\beta + e^{-\beta}}\, e^{-2i\ell\beta}\, d\beta \\ &= \int_{-\infty}^{\infty} \frac{1}{e^\beta + e^{-\beta}} \left(\cos\nu_a \frac{d}{d\beta} + i\sin\nu_a \right) e^{-2i\ell\beta}\, d\beta \\ &= \int_{-\infty}^{\infty} \frac{1}{e^\beta + e^{-\beta}} \Big(-2i\ell\cos\nu_a + i\sin\nu_a \Big) e^{-2i\ell\beta}\, d\beta , \end{aligned}$$

showing that the integral is obtained from the earlier integral (3.34) if one only replaces the prefactor ℓ by $\widetilde{\ell}$ given by

$$\widetilde{\ell}_a := \ell \cos \nu_a - \frac{\sin \nu_a}{2} . \tag{3.50}$$

The same method also applies to the integral (3.35) and again it amounts to the replacement (3.50). We conclude that the matrix in (3.37) ought to be replaced by the two matrices

$$\widehat{S}^a_{\mathfrak{R}}(\ell) = \frac{\widetilde{\ell}_a}{\pi m} \begin{pmatrix} \dfrac{1}{1+e^{-2\pi\ell}} & -\dfrac{i}{2\cosh(\pi\ell)} \\ \dfrac{i}{2\cosh(\pi\ell)} & \dfrac{1}{1+e^{2\pi\ell}} \end{pmatrix} .$$

These matrices have the eigenvalues

$$\lambda = 0 \qquad \text{and} \qquad \lambda = \frac{\widetilde{\ell}_a}{\pi m} .$$

As a consequence, the analogue of Theorem 3.3.2 is the following statement:

Theorem 3.5.1. *After separating the y- and z-dependence by the plane wave ansatz* (3.45)*, the fermionic signature operator S and the Hamiltonian H in Rindler coordinates satisfy the relations*

$$S = -\frac{H}{\pi \widetilde{m}} - \frac{1}{2\pi m \widetilde{m}} \gamma^0 \gamma^1 \Big(\gamma^2 \partial_y + \gamma^3 \partial_z \Big) \tag{3.51}$$

with $\widetilde{m}$ according to (3.46)*.*

Proof. Considering again a Lorentz boost, just as in the proof of Theorem 3.3.2 we find that $H = -\ell$. Therefore, considering as in (3.49) the situation that $k_z = 0$ and $k_y > 0$, we obtain on the first and fourth spinor components that

$$S_a = -\frac{H}{\pi m} \cos \nu_a - \frac{\sin \nu_a}{2\pi m}$$

with $a = 1$. Similarly, on the second and third spinor components, the same formula holds with $a = -1$. Using (3.48), we can simplify these equations to

$$S_a = -\frac{H}{\pi \widetilde{m}} - \frac{a k_y}{2\pi m \widetilde{m}} .$$

By direct computation, one verifies that the operator

$$\gamma^0 \gamma^1 \Big(\gamma^2 \partial_y + \gamma^3 \partial_z \Big)$$

has an eigenvalue k_y, and the corresponding eigenspace is the subspace spanned by the first and fourth spinor components. Likewise, the subspace spanned by the second and third spinor components is an eigenspace for the eigenvalue $-k_y$. This proves (3.51) for the case $k_z = 0$ and $k_y > 0$. The general case follows immediately because of the operator (3.51) is invariant under rotations in the yz-plane. ■

Remark 3.5.1. *The separation of the y- and z-dependence could be described more mathematically by a Fourier transform $\psi(t,x,y,z) \mapsto \widetilde{\psi}(t,x,k_y,k_z)$, being a unitary transformation between corresponding Hilbert spaces. Since this procedure is very similar to the one at the beginning of Section 3.2.2, we leave the details to the reader. Carrying out this procedure, the factors $1/\tilde{m}$ become multiplication operators in momentum space -see* (3.46)*. As a by-product, in position space, these operators are nonlocal in the variables y and z.*

Applying the constructions outlined in Section 3.4, we obtain again quasifree quantum states. However, these states are different from the Fulling-Rindler vacuum and the thermal states as obtained in Corollary 3.4.1 . The physical interpretation of these new states is still under investigation, as well as if the Hadamard condition is satisfied.

CHAPTER 4

FP-States on Spacetimes with Mass Oscillation Properties

As we saw in the previous chapter, in Rindler coordinates the massive Dirac equation reads

$$i\partial_\tau \psi = H\psi \,,$$

where H is the Hamiltonian, *i.e.,* an elliptic, self-adjoint operator on the space of solutions $\mathscr{H}_{\mathfrak{R}}$. The eigenvalues of H could be interpreted as the frequencies of a solution $\psi(\tau,\rho)$ and its sign gives a splitting of the solution space into two subspaces, usually referred to as the positive and negative energy subspaces. This frequency splitting is important for the physical interpretation of the massive Dirac equation and for the construction of a corresponding quantum field theory. To wit choosing the vacuum state in agreement with the frequency splitting -see Corollary 3.4.1 , it is possible to reinterpret the negative-energy solutions in terms of antiparticle states. The plane-wave solutions of positive and negative frequencies are then identified with creation and annihilation operators, respectively, which by acting on the vacuum state generate the whole Fock space. The above frequency splitting can still be used in stationary spacetimes, *i.e.* if a timelike Killing field is present. However, in *generic spacetimes* or in the presence of a *time-dependent* external potential, one does not have a natural frequency splitting. A common interpretation of this fact is that there is no distinguished ground state and that the notion particles and anti-particles depend on the observer. Nonetheless, the construction of the *fermionic projector* as carried out non-perturbatively in [FiRe15, FiRe16] does give rise to a canonical splitting of the solution space of the massive Dirac equation into two subspaces even in generic spacetimes. This also suggests that, mimicking the construction for the usual frequency splitting, there should be a canonical Fock state of the corresponding quantum field theory, even without assuming a Killing symmetry. One of the goals of this chapter is to construct this distinguished Fock state. To achieve our goal, we fix M to be a four-dimensional and globally hyperbolic spacetime.
The results of this chapter have already appeared as preprint in [FMR16a, DM16].

4.1 The Fermionic Projector

In the previous chapter, taking advantages of Lemma 3.2.1, we expressed the spacetime inner product (3.5) in terms of the scalar product (3.3) using the signature operator S. The above construction fails in generic spacetimes because the time integral in (3.5) will in general not be bounded or even diverge. In [FiRe16] it was designed a possible way out for four-dimensional spacetimes[1]. Let us consider families of solutions $\underline{\psi} := (\psi_m)_{m\in I}$ of the massive Dirac equation with the mass parameter m varying in an open interval I. From now, we will denote the Hilbert space of a solutions of the massive Dirac equation for a fixed m as $\mathscr{H}_m$. We need to assume that I does not contain the origin because this method does not apply to the massless case ($m = 0$) [2]. By symmetry, it suffices to consider positive masses. Thus we choose

$$I := (m_L, m_R) \subset \mathbb{R} \qquad \text{with parameters } m_L, m_R > 0\,.$$

A priory it is not obvious that these space is not empty. The reason for doubting about the existence of such function is related to the fact that one cannot consider simply a solution to the massive Dirac equation and let the mass m which appears in such solution vary. However, the next corollary shows that this is not the case.

Corollary 4.1.1. *Let M be a given spacetime. Then there exists always a family of solutions $\underline{\psi}$ of the massive Dirac equation in the class $C^\infty_{sc,c}(M \times I, \mathbb{C}^4)$ of smooth solutions with spatially compact support in a given spacetime M which depend smoothly on m and vanish identically for m outside a compact subset of I.*

Proof. The following construction shows that within this class, there are families $\underline{\psi}$ such that for every $m \in I$, the solution ψ is a solution of the massive Dirac equation (2.9): Let $\underline{\psi}_0 \in C^\infty_{sc,c}(\Sigma_{t_0} \times I, \mathbb{C}^4)$ be a family of smooth and compactly supported functions on $\Sigma_{t_0} \times I$, for example of the form $\underline{\psi}_0 = \eta(m)\,\chi(x)$ with $\eta \in C^\infty_c(I)$ and $\chi \in C^\infty_c(\Sigma, \mathbb{C}^4)$. Solving for every $m \in I$ the Cauchy problem

$$\begin{cases} (i\nabla_{(s)} - m)\underline{\psi} = 0 \\ \underline{\psi}|_{\Sigma_{t_0}} = \underline{\psi}_0 \end{cases} \tag{4.1}$$

we obtain a family $\underline{\psi}$ of solutions of the massive Dirac equation for a variable mass parameter $m \in I$ in the desired class $C^\infty_{sc,c}(M \times I, \mathbb{C}^4)$. ■

Next, we endow the $\mathbb{C}$-vector space made of those families of solutions with the following scalar product

$$\left(\underline{\psi} \mid \underline{\tilde{\psi}}\right)_\oplus := \int_I \left(\psi_m \mid \tilde{\psi}_m\right)_m dm, \tag{4.2}$$

where dm is the Lebesgue measure and $(\cdot \mid \cdot)_m$ denotes the scalar product in the Hilbert space $\mathscr{H}_m$, which involves integration over Σ. Taking the completion of such space we obtain the Hilbert space $\mathscr{H}^\oplus$. We denote with $\|\cdot\|_\oplus$ the norm of $\mathscr{H}^\oplus$ and $\|\cdot\|_m$ the norm of $\mathscr{H}_m$.

Remark 4.1.1. *The Hilbert space $\mathscr{H}^\oplus$ contains measurable sequences $\underline{\psi} = (\psi_m)_{m\in I}$ such that $\psi_m \in \mathscr{H}_m$ for almost all $m \in I$ and $\psi_m|_\Sigma$ is square integrable on any Cauchy surface Σ.*

[1] We expect that all these results can be generalised to a $n+1$-dimensional spin Lorentzian manifold with $n > 3$.

[2] This condition seems not to bear a physical restriction because all known Fermions in nature have a non-zero rest mass.

Remark 4.1.2. *In $\mathscr{H}^\oplus$ we can distinguish two different dense subspaces:*

- *The subspace made of sequences $\underline{\psi}^\bullet = (\psi^\bullet_m)_{m\in I}$ such that $\psi^\bullet_m \in \mathrm{Sol}(\mathscr{D}_m)$ for almost all $m \in I$;*
- *The subspace made of sequences $\underline{\psi} = (\psi_m)_{m\in I}$ which are compact in the mass parameter.*

We denote the first subspace with $\mathrm{Sol}^\oplus$ and the second one with $\mathscr{H}^\infty := C^\infty_{sc,c}(M \times I, \mathbb{C}^4) \cap \mathscr{H}^\oplus$.

On $\mathscr{H}^\infty$, we introduce the operator that integrates over the mass m.

Definition 4.1.1. *We call* ***smearing operator*** $\mathfrak{p}$ *the operator defined by*

$$\mathfrak{p}: \mathscr{H}^\infty \to \Gamma_{sc}(SM), \qquad \mathfrak{p}\underline{\psi} := \int_I \psi_m \, dm.$$

Remark 4.1.3. *Notice that, even if $\underline{\psi}^\bullet \in \mathrm{Sol}^\oplus$, $\mathfrak{p}\underline{\psi}^\bullet \notin \mathscr{H}_m$ regardless of the choice of m.*

The smearing operator plays a central role in the construction: First of all it allows to construct the sequilinear form

$$N : D(N) \subseteq \mathscr{H}^\infty \times \mathscr{H}^\infty \to \mathbb{C}, \qquad N(\underline{\psi}, \underline{\tilde{\psi}}) := \left\langle \mathfrak{p}\underline{\psi} \mid \mathfrak{p}\underline{\tilde{\psi}} \right\rangle,$$

where $\langle \mid \rangle$ is the spacetime inner product defined in (2.8). In addition

$$D(N) := \left\{ (\underline{\psi}, \underline{\tilde{\psi}}) \in \mathscr{H}^\oplus \times \mathscr{H}^\oplus \;\middle|\; \left\langle \underline{\psi} \mid \underline{\tilde{\psi}} \right\rangle \text{ exists finite} \right\}.$$

Secondly, it generates a decay of the solution, making possible that the time integral converges.

Before introducing on $\mathscr{H}^\infty$ a condition similar to (3.7), let us endow $\mathscr{H}^\oplus$ with a linear self-adjoint operator which multiplies by m

$$T : \mathscr{H}^\oplus \to \mathscr{H}^\oplus \qquad (T\underline{\psi}) := (m\,\psi_m)_{m\in I}.$$

It is a symmetric operator, and it is bounded because the interval I is, *i.e.*

$$T^* = T \in \mathscr{B}(\mathscr{H}).$$

In addition T preserves the support property:

$$T\big|_{\mathscr{H}^\infty} : \mathscr{H}^\infty \to \mathscr{H}^\infty.$$

Definition 4.1.2. *The Dirac operator $\mathscr{D}_m$ on M has the* ***weak mass oscillation property*** *(WMOP) in the interval I with domain $\mathscr{H}^\infty$ if :*

(i) For any $\underline{\psi} \in \mathscr{H}^\infty$ there exists a constant $C = C(\underline{\psi}) > 0$ such that

$$\left| \left\langle \underline{\psi} \mid \underline{\tilde{\psi}} \right\rangle \right| \le C(\underline{\psi}) \, \|\underline{\tilde{\psi}}\|_\oplus \qquad \forall \underline{\tilde{\psi}} \in \mathscr{H}^\infty; \tag{4.3}$$

(ii) It holds

$$\left\langle \mathfrak{p}T\underline{\psi} \mid \mathfrak{p}\underline{\tilde{\psi}} \right\rangle = \left\langle \mathfrak{p}\underline{\psi} \mid \mathfrak{p}T\underline{\psi} \right\rangle \qquad \forall \underline{\psi}, \underline{\tilde{\psi}} \in \mathscr{H}^\infty. \tag{4.4}$$

Clearly, in a given spacetime one must verify if the assumptions in this definition are satisfied. In view of the inequality (4.3), every $\underline{\psi} \in \mathscr{H}^\infty$ gives rise to a bounded linear functional on $\mathscr{H}^\infty$. Since $\mathscr{H}^\infty$ is dense in $\mathscr{H}^\oplus$, this linear functional can be uniquely extended to $\mathscr{H}^\oplus$ by continuity. The Riesz representation theorem allows us to represent this linear functional by a vector $u \in \mathscr{H}^\oplus$, *i.e.*

$$\left(\underline{\psi} \mid u\right)_\oplus = \left\langle \mathfrak{p}\underline{\psi} \mid \mathfrak{p}\underline{\tilde{\psi}}\right\rangle \qquad \forall \underline{\psi} \in \mathscr{H}^\oplus$$

Varying $\underline{\tilde{\psi}}$, we obtain the linear mapping

$$S : \mathscr{H}^\infty \to \mathscr{H}^\oplus, \qquad \left\langle \mathfrak{p}\underline{\psi} \mid \mathfrak{p}\underline{\tilde{\psi}}\right\rangle = \left(\underline{\psi} \mid S\underline{\tilde{\psi}}\right)_\oplus .$$

for any $\underline{\psi} \in \mathscr{H}^\oplus$. This operator is symmetric because

$$\left(S\underline{\psi} \mid \underline{\tilde{\psi}}\right)_\oplus = \left\langle \mathfrak{p}\underline{\psi} \mid \mathfrak{p}\underline{\tilde{\psi}}\right\rangle = \left(\underline{\psi} \mid S\underline{\tilde{\psi}}\right)_\oplus .$$

Moreover, equation (4.4) implies that the operators S and T commute,

$$ST = TS : \mathscr{H}^\infty \to \mathscr{H}^\oplus .$$

Up to now, we are not able to define a self-adjoint operator on $\mathscr{H}_m$. Therefore a stronger condition is needed.

Definition 4.1.3. *The Dirac operator $\mathscr{D}_m$ on M has the* ***strong mass oscillation property*** *(SMOP) in the interval I with domain $\mathscr{H}^\infty$ if the there exists a constant $C > 0$ such that*

$$\left|\left\langle \mathfrak{p}\underline{\psi} \mid \mathfrak{p}\underline{\tilde{\psi}}\right\rangle\right| \leq C \int_I \|\psi_m\|_m \|\tilde{\psi}_m\|_m dm \tag{4.5}$$

for any $\underline{\psi}, \underline{\tilde{\psi}} \in \mathscr{H}^\infty$.

The SMOP plays a crucial role, since it defines uniquely a family of bounded, symmetric operators $S_m \in \mathscr{H}_m$ for every $m \in I$.

Theorem 4.1.1. *The following statements are equivalent:*

(i) The SMOP holds.

(ii) There is a constant $C > 0$ such that for all for any $\underline{\psi}$, $\underline{\tilde{\psi}} \in \mathscr{H}^\infty$ the following two relations hold:

$$\begin{aligned} \left\langle \mathfrak{p}T\underline{\psi} \mid \mathfrak{p}\underline{\tilde{\psi}}\right\rangle &= \left\langle \mathfrak{p}\underline{\psi} \mid \mathfrak{p}T\underline{\psi}\right\rangle \\ \left|\left\langle \mathfrak{p}\underline{\psi} \mid \mathfrak{p}\underline{\tilde{\psi}}\right\rangle\right| &\leq C\|\underline{\psi}\|_\oplus \|\underline{\tilde{\psi}}\|_\oplus . \end{aligned} \tag{4.6}$$

(iii) There exists a family of linear operator $(S_m)_{m\in I}$, where each S_m acts on $\mathscr{H}_m$ as a self-adjoint linear bounded operator, such that

$$\sup_{m\in I} |S_m\|_m < +\infty \tag{4.7a}$$

$$\left(\underline{\psi} \mid S\underline{\tilde{\psi}}\right)_\oplus = \int_I \left(\psi_m \mid S_m\tilde{\psi}_m\right)_m dm \tag{4.7b}$$

for any $\underline{\psi}, \underline{\widetilde{\psi}} \in \mathscr{H}^\infty$. *Moreover, requiring that*

$$m \mapsto \big(\psi_m \mid S_m \widetilde{\psi}\big)_m \text{ is continuous}$$

the family $(S_m)_{m\in I}$ *is unique and the construction does not depend on the choice of the interval* I.

For the technical details, we refer to Theorem 4.2 and to Proposition 4.4 in [FiRe16]. As a direct consequence of (4.6), the SMOP implies the WMOP. As we already anticipated, using formula (4.7b) we can define a bounded, symmetric operator S_m on $\mathscr{H}_m$ is defined.

Definition 4.1.4. *Using the spectral calculus, we define the* ***fermionic projector*** *to be*

$$\Pi_{S_m} := \chi_{(0,\infty)}(S_m) = \int_\sigma \chi(\lambda)\, dE_\lambda : \mathscr{H}_m \to \mathscr{H}_m\,,$$

where χ *is the Heaviside step function, while* dE_λ *and* σ *are respectively the spectral measure and the spectrum of* S_m

Remark 4.1.4. *Let us stress, that out of the fermionic projector, we can construct a quasifree state on the algebra of Dirac fields. This can be accomplished by defining the operator* $P := \Pi_{S_m} \oplus (Id_{\mathscr{H}} - A\Pi_{S_m}A^{-1})$ *on* $\mathscr{H}_m$*, where* A *is the adjunction and* $\mathscr{H}_{\mathscr{H}}$ *is defined in Section 2.4. Applying Corollary 2.4.2, we obtain a quasifree state denoted by* ω_{FP}*. We refer to it as* ***FP state***.

4.1.1 Minkowski Spacetime and the Mass Oscillation Properties

When the Dirac operator satisfies the SMOP, the construction of the FP states can be thought of as a generalisation of the frequency splitting for the Dirac Hamiltonian. Moreover, on Minkowski spacetime, the decomposition of the solution space into the positive and negative spectral subspaces of the fermionic signature operator reduces to the usual frequency splitting.

Proposition 4.1.1. *The Dirac operator in Minkowski spacetime has the strong mass oscillation property and the FP state coincides with the vacuum state.*

Proof. Using the Fourier transform, a solution to the massive Dirac equation in Minkowski spacetime reads

$$\psi(q) = \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4}\, \delta(p^2 - m^2)\epsilon(\omega)(\gamma^\mu p_\mu + m)\gamma^0 \widehat{\psi}^0_m(\vec{k})\, e^{-ip^\mu q_\mu}\,,$$

where ϵ is the sign function, $q = (t, \vec{x})$, $p = (\omega, \vec{k})$ and $\widehat{\psi}^0_m(\vec{k})$ denotes the spatial Fourier transform of $\psi_m|_{t=0}$. This is a distribution supported on the mass shell and it is not square integrable over $\mathbb{R}^4$. Solving the Cauchy problem (4.1), we obtain a family of solutions $\underline{\psi} = (\psi_m)_{m\in I} \in \mathscr{H}^\infty$. Integrating over m, we have

$$(\mathfrak{p}\psi)(k) = 2\pi\, \chi_I(m)\, \frac{1}{2m}\, \epsilon(\omega^0)(\not{k} + m)\gamma^0 \widehat{\psi}^0_m(\vec{k})\Big|_{m=\sqrt{p^2}}, \tag{4.8}$$

where m now is a function of the momentum variables. Since the function $\psi_m|_{t=0}$ is compactly supported and smooth in the spatial variables, its Fourier transform $\widehat{\psi}^0_m(\vec{k})$ has rapid decay. This shows that the function (4.8) is indeed square integrable. Using Plancherel's theorem, we see that condition (ii) in Definition 4.1.2 is satisfied. Moreover, the operator T is simply the operator of multiplication by $\sqrt{k^2}$, so that condition (i) obviously holds. This shows again the weak mass oscillation property.

In order to prove the strong mass oscillation property, we need to compute the inner product $\langle \mathfrak{p}\underline{\psi} \mid \mathfrak{p}\underline{\tilde{\psi}} \rangle$. To this end, we first write this inner product in momentum space as

$$\begin{aligned}\langle \mathfrak{p}\underline{\psi} \mid \mathfrak{p}\underline{\tilde{\psi}} \rangle &= \int \frac{d^4p}{(2\pi)^4} 4\pi^2\, \chi_I(m)\, \frac{1}{4m^2} \prec (\gamma^\mu p_\mu + m)\gamma^0 \hat{\psi}^0_m(\vec{k}) \mid (\gamma^\mu p_\mu + m)\gamma^0 \hat{\tilde{\psi}}^0_m(\vec{k}) \succ \Big|_{m=\sqrt{p^2}} \\ &= \int \frac{d^4k}{4\pi^2}\, \chi_I(m)\, \frac{1}{2m} \prec \gamma^0 \hat{\psi}^0_m(\vec{k}) \mid (\gamma^\mu p_\mu + m)\gamma^0 \hat{\tilde{\psi}}^0_m(\vec{k}) \succ \Big|_{m=\sqrt{p^2}} .\end{aligned}$$

Reparametrizing the ω^0-integral as an integral over m, we obtain

$$\langle \mathfrak{p}\underline{\psi} \mid \mathfrak{p}\underline{\tilde{\psi}} \rangle = \frac{1}{4\pi^2} \int_I dm \int_{\mathbb{R}^3} \frac{d^3k}{2\,|\omega^0|} \prec \gamma^0 \hat{\psi}^0_m(\vec{k}) \mid (\gamma^\mu p_\mu + m)\gamma^0 \hat{\tilde{\psi}}^0_m(\vec{k}) \succ \Big|_{k^0=\pm\sqrt{|\vec{k}|^2+m^2}} .$$

Estimating the inner product with the Schwartz inequality and applying Plancherel's theorem, one finds

$$\left| \langle \mathfrak{p}\underline{\psi} \mid \mathfrak{p}\underline{\tilde{\psi}} \rangle \right| \le \frac{1}{4\pi^2} \int_I dm \int_{\mathbb{R}^3} \|\hat{\psi}^0_m(\vec{k})\|\, \|\hat{\tilde{\psi}}^0_m(\vec{k})\|\, d^3k \le 2\pi \int_I \|\psi_m\|_m\, \|\phi_m\|_m\, dm .$$

Thus the inequality (4.5) holds. Applying Plancherel's theorem and using (4.2), we conclude that

$$\langle \mathfrak{p}\underline{\psi} \mid \mathfrak{p}\underline{\tilde{\psi}} \rangle = \int_I \left(\psi_m \mid S_m(\vec{k}) \tilde{\psi}^0_m \right)_m dm , \tag{4.9}$$

where

$$S_m(\vec{k}) := \sum_{\omega=\pm\omega(\vec{k})} \frac{\gamma^\mu p_\mu + m}{2\,\omega(\vec{k})} \gamma^0 = \frac{\vec{k}\vec{\gamma} + m}{\omega(\vec{k})} \gamma^0 .$$

Comparing (4.9) with (4.7b), one sees that the matrix $S_m(\vec{k})$ is indeed the fermionic signature operator, considered as a multiplication operator in momentum space. By direct computation, one verifies that the matrix $S_m(\vec{k})$ has eigenvalues ± 1 and $\Pi_{S_m} \equiv \chi_{(0,\infty)}(H)$, being H the Dirac Hamiltonian. ■

4.1.2 Rindler Spacetime and the Mass Oscillation Properties

We prove now that, while the SMOP holds true on Minkowski spacetime, it does not in Rindler. This is due to the failure of the equality

$$\langle \mathfrak{p}T\underline{\psi} \mid \mathfrak{p}\underline{\tilde{\psi}} \rangle = \langle \mathfrak{p}\underline{\psi} \mid \mathfrak{p}T\underline{\psi} \rangle \tag{4.10}$$

for the presence of a horizon. Thus none of the MOPs introduced in Definitions 4.1.2 and 4.1.3 can be satisfied.

Proposition 4.1.2. *The Dirac operator in Rindler spacetime does not have the weak mass oscillation property.*

Proof. Using the results of Lemma 3.1.1, a spacelike compact solution ψ of the massive Dirac equation in Rindler spacetime can be obtained by restriction of a spacelike compact solution Ψ of the same equation on the whole Minkowski spacetime. Moreover the norm of ψ in the space $\mathscr{H}_{\mathscr{R}}$ coincides with the norm of Ψ on $\mathscr{H}_{\mathscr{M}}$, i.e. $\|\psi\|_{\mathscr{R}} = \|\Psi\|_{\mathscr{M}}$. We can write $\psi = \chi_{\mathscr{R}}\Psi$, being $\chi_{\mathscr{R}}$ the characteristic function of $\mathscr{R}$. Solving the Cauchy problem (4.1), we denote the families of solutions in Rindler and Minkowski with $\underline{\psi}$ and $\underline{\Psi}$ respectively. The sesquilinear form $N_{\mathscr{R}}$ reads

$$N_{\mathscr{R}}(\underline{\psi},\underline{\tilde{\psi}}) = \left\langle \mathfrak{p}\underline{\psi} \mid \mathfrak{p}\underline{\tilde{\psi}}\right\rangle_{\mathscr{R}} = \int_{\mathscr{R}} \prec \mathfrak{p}\underline{\psi} \mid \mathfrak{p}\underline{\tilde{\psi}} \succ d\mu_g, \tag{4.11}$$

being $d\mu_g$ the induce volume measure on $\mathscr{R}$ and being

$$D(N_{\mathscr{R}}) := \left\{(\underline{\psi},\underline{\tilde{\psi}}) \in \mathscr{H}^{\infty}_{\mathscr{R}} \times \mathscr{H}^{\infty}_{\mathscr{R}} \middle| \left\langle \mathfrak{p}\underline{\psi} \mid \mathfrak{p}\underline{\tilde{\psi}}\right\rangle \text{exists finite}\right\} \subseteq \mathscr{H}^{\infty}_{\mathscr{R}} \times \mathscr{H}^{\infty}_{\mathscr{R}}.$$

We prove now that condition (4.10) fails. Using $T\underline{\psi} = i\gamma^{\mu}\partial_{\mu}\underline{\psi}$, partial integration gives a boundary term

$$N_{\mathscr{R}}(T\underline{\psi},\underline{\tilde{\psi}}) - N_{\mathscr{R}}(\underline{\psi},T\underline{\tilde{\psi}}) = \int_{\partial\mathscr{R}} \prec \mathfrak{p}\underline{\Psi} \mid \mathfrak{p}\underline{\tilde{\Psi}} \succ |_{\partial\mathscr{R}},$$

where $\partial\mathscr{R} := \{(t,x,y,z) \in \mathscr{M} \mid |t| = x\}$. To show that this latter contribution does not vanish in general, we make the change of coordinates $(t,x) \mapsto (\tau,\rho)$ and we observe that

$$\begin{aligned}\int_{\partial\mathscr{R}} \prec \mathfrak{p}\underline{\Psi} \mid \mathfrak{p}\underline{\tilde{\Psi}} \succ |_{\partial\mathscr{R}} = i\int_I dm \int_I d\tilde{m} \int_{\mathbb{R}^2} dx_{\perp} \int_0^{+\infty} d\tau \Big[&(\overline{\Psi}_m(\gamma^1-\gamma^0)\tilde{\Psi}_{\tilde{m}})(\tau,\tau,x_{\perp}) \\ &+(\overline{\Psi}_m(\gamma^1+\gamma^0)\tilde{\Psi}_{\tilde{m}})(-\tau,\tau,x_{\perp})\Big],\end{aligned} \tag{4.12}$$

where $\overline{(\cdot)}$ is the complex conjugation and $x_{\perp} = (y,z)$. Note that, since Ψ_m, $\tilde{\Psi}_{\tilde{m}}$ are solutions on $\mathscr{M}$, the integrand is strictly supported for $s \in (0,+\infty)$, therefore we can extend the integral over the whole real line. Without loss of generality, we choose Ψ_m and $\tilde{\Psi}_{\tilde{m}}$ as positive frequency solutions, namely

$$\begin{aligned}\Psi_m(t,x,x_{\perp}) &= \int dk_{\perp}\int dk c_{+}(k,k_{\perp},m)e^{-i\omega t}e^{ikx}e^{ik_{\perp}x_{\perp}}, \\ \tilde{\Psi}_{\tilde{m}}(t,x,x_{\perp}) &= \int dk_{\perp}\int dk \tilde{c}_{+}(k,k_{\perp},\tilde{m})e^{-i\tilde{\omega} t}e^{ikx}e^{ik_{\perp}x_{\perp}}\end{aligned} \tag{4.13}$$

with $\omega^2 = k^2 + |k_{\perp}|^2 + m^2$ (resp. $\tilde{\omega}^2 = k^2 + |k_{\perp}|^2 + \tilde{m}^2$) and c_{+} (resp. $\tilde{c}_{+}$) a suitable smooth function in k which is compactly supported in the mass variable m (resp. $\tilde{m}$). For simplicity, we also assume that c_{+} and $\tilde{c}_{+}$ are symmetric in the k variable. Later on, we will argue that the dependence on the mass parameter can be chosen in such a way that the integral in (4.12) does not vanish. Using the Fourier representation (4.13) in (4.12), we find

$$\begin{aligned}&\int_{\mathbb{R}^2} dx_{\perp}\int_{\mathbb{R}} d\tau \left[(\overline{\Psi}_m(\gamma^1-\gamma^0)\tilde{\Psi}_{\tilde{m}})(\tau,\tau,x_{\perp}) + (\overline{\Psi}_m(\gamma^1+\gamma^0)\tilde{\Psi}_{\tilde{m}})(-\tau,\tau,x_{\perp})\right] \\ &\quad= \int_{\mathbb{R}^2} dk_{\perp}\int_{\mathbb{R}} dk \int_{\mathbb{R}} dp \left[\delta(k_{-}-p_{-})\overline{c_{+}}(k,k_{\perp},m)(\gamma^1-\gamma^0)\tilde{c}_{+}(p,k_{\perp},\tilde{m})\right. \\ &\qquad\left. +\delta(k_{+}-p_{+})\overline{c_{+}}(k,k_{\perp},m)(\gamma^1+\gamma^0)\tilde{c}_{+}(p,k_{\perp},\tilde{m})\right],\end{aligned} \tag{4.14}$$

where $k_\pm = \omega \pm k$, $p_\pm = \widetilde{\omega} \pm p$. Changing variables and exploiting the symmetry of c_+, $\widetilde{c_+}$ in k we are lead to

$$(4.14) = 2\int_0^{+\infty} dk\; r(k,k_\perp,m)r(k,k_\perp,\widetilde{m})\overline{c_+}\left(\frac{k^2-m^2(k_\perp)}{2k},m\right)\gamma^1\widetilde{c}_+\left(\frac{k^2-\widetilde{m}^2(k_\perp)}{2k},\widetilde{m}\right),$$

where $m^2(k_\perp) = m^2 + |k_\perp|^2$ and $r(k,k_\perp,m) = \left(k^2+m^2(k_\perp)\right)/(2k^2)$. It is then enough to choose $c_+, \widetilde{c}_+$ in such a way that $r(k,k_\perp,m)\overline{c_+}\Big(\left(k^2-m^2(k_\perp)\right)/(2k),m\Big)$ is the total derivative in the mass parameter m of a smooth function in m which, at $m = \sup I$, is equal to a positive fast decreasing function in k, while it vanishes at $m = \inf I$. The integral is thus non-vanishing. ■

4.2 New Classes of Fermionic Projectors

The abstract construction in spacetimes as given in Section 4.1 opens up the research program to explore the fermionic signature operator in various spacetimes and to verify whether the resulting FP states are Hadamard. So far, the fermionic signature operator has been studied in the examples of ultrastatic spacetimes and of de Sitter spacetime [FiRe16] and of Minkowski space in the presence of an external potential [FMR16a, FiRe17]. As the first example involving a horizon, we considered Rindler spacetime in Subsection 4.1.2. Here the methods of Section 4.1 do not apply. The reason is that the mass oscillation properties do not hold due to boundary contributions on the horizon. Nonetheless, Rindler spacetime is of physical interest in view of the Unruh effect, which is closely related to the Hawking effect in black hole geometries as we saw in Chapter 3. It is desirable to have a weaker requirement for implementing the procedure: We shall show below that this can be performed in a non-canonical way. The key idea is to avoid the SMOP by constructing a continuous immersion $\mathscr{H}_m \hookrightarrow \mathscr{H}^\oplus$ through a suitable bounded map $\mathfrak{R}$. Composing the smearing operator $\mathfrak{p}$ with $\mathfrak{R}$ would lead to a modified version of both SMOP and WMOP. In particular, the modified MOPs are now formulated directly on $\mathscr{H}_m$. Thus it will be enough to check the last propriety to define a pure quasifree state on the algebra of observables for Dirac fields $\mathfrak{F}(M)$.

Let us introduce the embeddings $\mathfrak{R} : \mathscr{H}_m \to \mathscr{H}^\oplus$ which are nothing but a direct sum of "Møller type" maps. These are well known in the literature, and they allow to intertwine the dynamics of two Green hyperbolic operators differing by a smooth potential. The Møller map was used by Peierls [Pe52] as a general procedure to define the Poisson brackets for the algebra of observables. Results on the existence of Møller operators can be found in [DF03, BF09, DHP17] and references therein. With this in mind, let us consider two Cauchy surfaces $\Sigma_\pm$ such that Σ_+ lies in the future of Σ_-. Let $\rho^\pm \in C^\infty(\mathbb{R})$ be a non decreasing function such that $\rho^+|_{J^+(\Sigma_+)} = 1$, $\rho^+|_{J^-(\Sigma_-)} = 0$ and $\rho^- = 1-\rho^+$. For any $\psi \in \mathscr{H}_m$, we define $R_{\widetilde{m},m}\psi \in \mathscr{H}_{\widetilde{m}}$ as follows:

- First consider the unitary operator which maps the Cauchy data $\psi|_{\Sigma_-}$ to the counterpart on Σ_+ by evolving it via the dynamics ruled by the Dirac equation with mass $M := \widetilde{m}\rho^- + m\rho^+$;

- Second, define $R_{\widetilde{m},m}\psi$ as the solution of the Dirac equation with mass $\widetilde{m}$ and Cauchy data provided by those previously obtained on Σ_+.

The whole procedure can be described quite explicitly as the composition of the following maps:

$$R^+_{M,m} = \mathrm{Id} - E^+_M(\tilde{m} - m)\rho^+ , \qquad R^-_{\tilde{m},M} = \mathrm{Id} - E^-_{\tilde{m}}(m - \tilde{m})\rho^- ,$$

where E^+_M denotes the advanced Green operator for the massive Dirac equation with mass M .

Remark 4.2.1. *Note that* $(\tilde{m} - m)\rho^+$ *and* $(m - \tilde{m})\rho^-$ *are past and future compact respectively, thus the composition with* E^+_M*,* $E^-_{\tilde{m}}$ *is well defined -we refer to [Bä15] for more details.*

Definition 4.2.1. *We call* ***Møller-Dappiaggi operator*** *the densely defined, linear, unitary map*

$$R_{\tilde{m},m} = R^-_{\tilde{m},M} \circ R^+_{M,m} : \mathscr{H}_m \to \mathscr{H}_{\tilde{m}}.$$

Varying $\tilde{m}$ *in an interval* I*, we obtain the so-called* ***embedding operator****, namely the unitary map* $\mathfrak{R}_m : \mathscr{H}_m \hookrightarrow \mathscr{H}^{\oplus}$.

Remark 4.2.2. *The construction of* $R^{\pm}_{\tilde{m},m}$ *and* $R_{\tilde{m},m}$ *can be read in the framework of scattering theory, and amounts to consider a partial S-matrix, without performing the so-called adiabatic limit. The possibility of removing the cut-off dependence on* $\rho^{\pm}$ *in this, or any other, sense is not a goal of the present work.*

In view of the continuous dependence of the solutions of the massive Dirac equation on the parameter (see [Ta11]), the function $\tilde{m} \mapsto R_{\tilde{m},m}\psi$ is integrable in the sense required by the definition of $\mathscr{H}^{\oplus}$: Furthermore

$$\|\mathfrak{R}_m \psi\|^2_I = \int_I \|R_{\tilde{m},m}\psi\|^2_m d\tilde{m} = |I| \|\psi\|^2_m < \infty,$$

where $|I|$ is the length of the interval I, thus proving that $\mathfrak{R}_m\psi \in \mathscr{H}^{\oplus}$ and that $\mathfrak{R}_m$ is an almost isometric linear bounded operator from $\mathscr{H}_m$ to $\mathscr{H}^{\oplus}$, with $\|\mathfrak{R}_m\| = \sqrt{|I|}$.

Remark 4.2.3. *Notice that, for any* $\psi_m \in \mathscr{H}_m$*, the element* $\mathfrak{R}_m\psi$ *lies in* $\mathrm{Sol}^{\oplus}$ *-see Remark 4.1.2- but a priori not in* $\mathscr{H}^{\infty}$ *since we have no control on the support properties of* $\mathfrak{R}_m\psi$ *as a function of* $\tilde{m}$*. Thus, in order to make contact with the definitions of WMOP and SMOP, we localise* $\mathfrak{R}_m\psi$ *with an arbitrary smooth, compactly supported function* $\mathfrak{m} \in C^{\infty}_{\mathrm{c}}(I)$.

Using the Møller-Dappiaggi operator, we formulate the mass oscillation property directly on $\mathscr{H}_m$.

Definition 4.2.2. *The Dirac operator* $\mathscr{D}_m$ *on* M *has the* ***modified weak mass oscillation property*** *(mWMOP) in the interval* I *with domain* $\mathrm{Sol}(\mathscr{D}_m)$ *if, for any* $\mathfrak{m} \in C^{\infty}_{\mathrm{c}}(I)$ *and* $\psi_1 \in \mathrm{Sol}(\mathscr{D}_m)$*, there exists a constant* $C(\mathfrak{m}, \psi_1) > 0$ *such that*

$$|\langle \mathfrak{p}\mathfrak{m}\mathfrak{R}_m\psi_1 \mid \mathfrak{p}\mathfrak{m}\mathfrak{R}_m\psi_2\rangle| \le C(\mathfrak{m}, \psi_1)\|\psi_2\|_m \qquad \forall \psi_2 \in \mathrm{Sol}(\mathscr{D}_m). \tag{4.15}$$

Similarly, the Dirac operator $\mathscr{D}_m$ *on* M *has the* ***modified strong mass oscillation property*** *(mSMOP) in the interval* I *with domain* $\mathrm{Sol}(\mathscr{D}_m)$ *if for any* $\mathfrak{m} \in C^{\infty}_{\mathrm{c}}(I)$ *there exists a constant* $C(\mathfrak{m}) > 0$ *such that*

$$|\langle \mathfrak{p}\mathfrak{m}\mathfrak{R}_m\psi_1 \mid \mathfrak{p}\mathfrak{m}\mathfrak{R}_m\psi_2\rangle \le C(\mathfrak{m})\|\psi_1\|_m \|\psi_2\|_m \qquad \forall \psi_1, \psi_2 \in \mathrm{Sol}(\mathscr{D}_m). \tag{4.16}$$

Remark 4.2.4. *Comparing Definition 4.2.2 with Definitions 4.1.2 and 4.1.3, we avoid the commutation property* (4.4)*. This is a requirement used to show the equivalence between the different formulations of SMOP* (4.5) *and* (4.6) *and it plays no role in our construction.*

Corollary 4.2.1. *The modified mass oscillation properties are weaker requirement than the mass oscillation property, namely the following diagram holds*

$$\begin{array}{ccc} SMOP & \Longrightarrow & WMOP \\ \Downarrow & & \Downarrow \\ mSMOP & \Longrightarrow & mWMOP\,. \end{array}$$

Proof. Let be $\underline{\psi},\underline{\widetilde{\psi}} \in \mathscr{H}^\infty$ and $\psi_1, \psi_2 \in \mathrm{Sol}(\mathscr{D}_m)$. The horizontal arrows descend taking $C(\underline{\psi}) = C||\underline{\psi}||_\oplus$ in (4.3) and $C(\mathfrak{m},\psi_1) = C(\mathfrak{m})\|\psi_1\|_m$ in (4.15). In order to show that SMOP⇒mSMOP (and similarly for the weak properties), it is enough to substitute $\underline{\psi} = \mathfrak{m}\mathfrak{R}\psi_1$, $\underline{\widetilde{\psi}} = \mathfrak{m}\mathfrak{R}_m\psi_2$ in (4.6) and to use $\|\mathfrak{m}\mathfrak{R}_m\psi_1\|_\oplus = \|\mathfrak{m}\|_{L^2(I)}\|\psi_1\|_m$ [3]. This proves that the m-WMOP is a proper weaker requirement than both the SMOP and the WMOP. ■

We are now in the position to describe how to build a quasifree state from the modified MOPs. For the clarity of the presentation, we discuss first the case in which the mSMOP holds true. Eventually, we focus on the case where only the mWMOP holds true. We stress that the main point here is to provide a construction of a state which holds also in cases where the construction of [FiRe16] cannot be applied immediately.

Theorem 4.2.1. *If the mSMOP holds true, then for any $\mathfrak{m} \in C^\infty_c(I)$, there exists a unique self-adjoint operator $\mathfrak{S}: \mathscr{H}_m \to \mathscr{H}_m$, henceforth called* ***modified fermionic signature operator****, defined by*

$$\langle \mathfrak{p}\mathfrak{m}\mathfrak{R}\psi_1 \mid \mathfrak{p}\mathfrak{m}\mathfrak{R}\psi_2\rangle = (\psi_1 \mid \mathfrak{S}\psi_2) \tag{4.17}$$

for any $\psi_1 \in \mathrm{Sol}(\mathscr{D}_m)$ and $\psi_2 \in \mathscr{H}_m$. The spectral decomposition of $\mathfrak{S}$ yields a spectral projector

$$\Pi^+_{mFP} = \chi_{(0,\infty)}(\mathfrak{S}): \mathscr{H}_m \to \mathscr{H}_m \tag{4.18}$$

and hence a quasifree state $\omega_{FP}: \mathfrak{F}(M) \to \mathbb{C}$.

Proof. , If the mSMOP holds true then, for any $\mathfrak{m} \in C^\infty_c(I)$, by (4.16) we can define a linear bounded operator $\mathfrak{S}: \mathscr{H}_m \to \mathscr{H}_m$ via Riesz Theorem. Indeed (4.16) ensures that, for any $\psi_1 \in \mathrm{Sol}(\mathscr{D}_m)$, $\psi_2 \mapsto \langle \mathfrak{p}\mathfrak{m}\mathfrak{R}\psi_1 \mid \mathfrak{p}\mathfrak{m}\mathfrak{R}\psi_2\rangle$ is a densely defined continuous linear functional. After extension on $\mathscr{H}_m$ we can apply Riesz Theorem to obtain

$$\langle \mathfrak{p}\mathfrak{m}\mathfrak{R}\psi_1 \mid \mathfrak{p}\mathfrak{m}\mathfrak{R}\psi_2\rangle = (\psi_1 \mid \mathfrak{S}\psi_2) \qquad \forall\psi_1 \in \mathrm{Sol}(\mathscr{D}_m), \forall\psi_2 \in \mathscr{H}_m,$$

[3] With a little more effort, one may prove that SMOP implies that, there is a constant $C > 0$ such that, for any $\psi_1, \psi_2 \in \mathrm{Sol}(\mathscr{D}_m)$, it holds $|\langle \mathfrak{p}\mathfrak{R}_m\psi_1 \mid \mathfrak{p}\mathfrak{R}_m\psi_2\rangle| \le C\|\psi\|_m\|\phi\|_m$. Since this alternative modified SMOP plays no role in the following, we will stick ourselves with the "localized" definition, which allows a more direct comparison between MOPs and mMOPs.

where we already made explicit the linear dependence on ψ_1 of the element $\mathfrak{S}\psi_1 \in \mathscr{H}_m$. Thus, we have found the modified fermionic signature operator, namely a linear map $\mathfrak{S} : \mathrm{Sol}(\mathscr{D}_m) \to \mathscr{H}_m$, which is also symmetric by (4.17). Notice that this procedure only makes use of the bound on ψ_2: Hence it is valid also in the case of the mWMOP holding. In the case of the mSMOP, we can also conclude that $\mathfrak{S}$ is bounded, actually $\|\mathfrak{S}\| \le C(\mathfrak{m})$. We have thus a self-adjoint operator $\mathfrak{S} : \mathscr{H}_m \to \mathscr{H}_m$, whose spectral decomposition allows to define the projector $\Pi_{\mathrm{mFP}} = \chi_{(0,\infty)}(\mathfrak{S})$. From it we can construct a state, ω_{mFP} by applying Lemma 2.4.1, once we defined

$$Q := \Pi_{\mathrm{FP}} \oplus (I_{\mathscr{H}} - A\Pi_{\mathrm{FP}}A^{-1}),$$

where A is once again the adjunction . This completes the construction of the modified fermionic projector state in the case of the mSMOP. ■

We deal now with the case in which the mWMOP holds true but not the mSMOP.

Theorem 4.2.2. *If the mWMOP holds true, then there exists a densely defined, symmetric operator* $\mathfrak{S} : \mathscr{H}_m \to \mathscr{H}_m$. *Moreover we obtain a quasifree state* $\omega_{FP} : \mathfrak{F}(M) \to \mathbb{C}$ *out of the spectral projection* $\Pi_{mFP} : \mathscr{H}_m \to \mathscr{H}_m$ *defined by*

$$\Pi_{mFP} = \frac{1}{2}\int_{\sigma(\mathfrak{S}^2)} \lambda^{-\frac{1}{2}}(\mathfrak{S} + \lambda^{\frac{1}{2}})\, d\mu_\lambda \tag{4.19}$$

where dE_λ *and* $\sigma(\mathfrak{S}^2)$ *are respectively the spectral measure and the spectrum of* $\mathfrak{S}^2$.

Proof. Following the first part of the proof of Theorem 4.2.1, the mWMOP and Riesz representation Theorem allow us to introduce, for all choices of $\mathfrak{m} \in C^\infty_c(I)$, a densely defined symmetric linear operator $\mathfrak{S}$

$$\langle \mathfrak{pm}\mathfrak{R}\psi_1 \mid \mathfrak{pm}\mathfrak{R}\psi_2 \rangle = (\psi_1 \mid \mathfrak{S}\psi_2) \qquad \forall \psi_1 \in \mathrm{Sol}(\mathscr{D}_m), \forall \psi_2 \in \mathscr{H}_m.$$

However, this time we have a priori no boundedness condition on $\mathfrak{S}$. Nevertheless, we can still employ techniques similar to the ones used in [FiRe16] (cfr. Section 3.2) to obtain the projector Π_{mFP}. First we use the Friederich extension for $\mathfrak{S}^2$ (see [La02] for more details); subsequently we define a spectral projector of $\mathfrak{S}$ as

$$\Pi_{\mathrm{mFP}} := \chi(\mathfrak{S}) := \frac{1}{2\sqrt{\mathfrak{S}^2}}\left(\mathfrak{S} + \sqrt{\mathfrak{S}^2}\right) = \frac{1}{2}\int_{\sigma(\mathfrak{S}^2)} \lambda^{-\frac{1}{2}}(\mathfrak{S} + \lambda^{\frac{1}{2}})\, dE_\lambda,$$

where dE_λ and $\sigma(\mathfrak{S}^2)$ are respectively the spectral measure and the spectrum of $\mathfrak{S}^2$. As before, we obtain the quasifree state ω_{FP} by introducing $Q := \Pi_{\mathrm{FP}} \oplus (I_{\mathscr{H}} - \Upsilon\Pi_{\mathrm{FP}}\Upsilon)$ and by applying Lemma 2.4.1. ■

Remark 4.2.5. *Whenever the modified fermionic signature operator* $\mathfrak{S}$ *is also self-adjoint, formula* (4.19) *is reduced to* (4.18).

For completeness, it is worth to make a couple of observations on the above-described procedure. First, we make a connection with the construction of [FiRe16]; in particular, we discuss the relation between the operators S and $\mathfrak{S}$. For that, let assume that the SMOP holds true. For any $\psi_1, \psi_2 \in \mathrm{Sol}(\mathscr{D}_m)$ we thus have

$$\begin{aligned}(\psi_1 \mid \mathfrak{S}\psi_2) &= \langle \mathfrak{p}\mathfrak{m}\mathfrak{R}\psi_1 \mid \mathfrak{p}\mathfrak{m}\mathfrak{R}\psi_2\rangle \\ &= (\mathfrak{m}\mathfrak{R}\psi_1 \mid S\mathfrak{m}\mathfrak{R}\psi_2)_{I_m} \\ &= \left(\psi_1 \mid (\mathfrak{m}\mathfrak{R})^\dagger S\mathfrak{m}\mathfrak{R}\psi_2\right).\end{aligned}$$

Thus we find $\mathfrak{S} = (\mathfrak{m}\mathfrak{R})^\dagger S\mathfrak{m}\mathfrak{R}$ or explicitly

$$\mathfrak{S}\psi_2 = \int_I d\tilde{m} |\mathfrak{m}_{\tilde{m}}|^2 R^\dagger_{\tilde{m},m} S_{\tilde{m}} R_{\tilde{m},m}\psi_2.$$

Another relevant feature is the $\mathfrak{m}$-dependence. By construction, our procedure depends on the chosen mass cut-off $\mathfrak{m} \in C^\infty_c(I)$. Definition 4.2.2 allows to perform the construction of ω_{mFP} for any choice of $\mathfrak{m}$, but it does not fix any continuous dependence of this latter parameter. [4] As a consequence we do not expect, in general, to be able to take the limit $\mathfrak{m} \to 1$ which would remove the dependence on $\mathfrak{m}$. Finally, notice that for the whole construction of the fermionic signature operators S and $\mathfrak{S}$, we restricted our attention to the case of constant mass m. In the case of non-constant $m \in C^\infty(M,\mathbb{R})$ we can still successfully apply the intertwining operator $R_{\tilde{m},m}$, but it is not immediately clear what should be analogous to the space $\mathscr{H}^\oplus$. One may try to consider $\tilde{m} \in C^\infty(M,\mathbb{R}) \cap L^2(M,g)$, thus inducing a Gaussian measure on that space: The space $\mathscr{H}^\oplus$ may than be defined as in Section 4.1 with $d\tilde{m}$ substituted by the Gaussian measure.

4.2.1 Rindler Spacetime and the Modified Mass Oscillation Properties

Now we show that, despite the MOPs are not satisfied by the Dirac operator on the Rindler spacetime for the presence of the horizon $\partial\mathscr{R}$, the mWMOP holds true.

Proposition 4.2.1. *The Dirac operator in Rindler spacetime satisfies the modified weak mass oscillation property.*

Proof. As in the proof of Proposition 4.1.2, let us denote with $\underline{\psi}$ and $\underline{\Psi}$ families of solutions in Rindler and Minkowski spacetimes respectively and let $N_{\mathscr{R}} : D(N_{\mathscr{R}}) \to \mathbb{C}$ be the sesquilinear form (4.11). Exploiting the relation between $\underline{\psi}$ and $\underline{\Psi}$, we find

$$\begin{aligned}|N_{\mathscr{R}}(\underline{\psi},\underline{\tilde{\psi}})| &= \left|\int_{\mathscr{M}} \prec \mathfrak{p}\underline{\Psi} \mid \chi_{\mathscr{R}}\mathfrak{p}\underline{\tilde{\Psi}} \succ d\mu_g\right| \\ &= \int_{\mathbb{R}}\int_{\Sigma_t} \prec \mathfrak{p}\underline{\Psi} \mid (\gamma^0)^2\chi_{\mathscr{R}}\mathfrak{p}\underline{\tilde{\Psi}} \succ |_{\Sigma_t} d\mu_\Sigma dt \\ &= \left|\int_{\mathbb{R}} (\mathfrak{p}\underline{\Psi}|_{\Sigma_t} \mid \gamma^0\chi_{\mathscr{R}}\mathfrak{p}\underline{\tilde{\Psi}}|_{\Sigma_t})_t\, dt\right|,\end{aligned}$$

[4] One could modify the definition of both mWMOP and mSMOP in order to avoid the mass cut off $\mathfrak{m}$, but this would create additional difficulties in the comparison between the mMOPs and the original MOPs.

being $d\mu_g$ the induced volume measure on $\mathscr{M}$ and $\chi_{\mathscr{R}}$ the characteristic function of $\mathscr{R}$. In the last equality we have fixed the foliation of $\mathscr{M}$ in terms of the Cauchy hypersurfaces $\Sigma_t = \{t = \text{constant}\}$ (note that such hypersurfaces are not Cauchy hypersurfaces for $\mathscr{R}$). The scalar product $(\cdot \mid \cdot)_t$ is the (time dependent) scalar product on $L^2(\Sigma_t)$, formally equal to (2.7). However, note that $\big(\mathfrak{p}\underline{\Psi}|_{\Sigma_t} \mid \gamma^0 \mathbf{1}_{\mathscr{R}} \mathfrak{p}\underline{\widetilde{\Psi}}|_{\Sigma_t}\big)_t$ is not time independent since none of the functions involved is a solution of the massive Dirac equation. Nevertheless, for any $t \in \mathbb{R}$, both $\mathfrak{p}\underline{\Psi}|_{\Sigma_t}$ and $\chi_{\mathscr{R}}\mathfrak{p}\underline{\widetilde{\Psi}}|_{\Sigma_t}$ lie in $L^2(\Sigma_t)$. Thus, by applying the Schwartz and Hölder inequalities

$$|N_{\mathscr{R}}(\underline{\psi}, \underline{\widetilde{\psi}})| \leq \int_{\mathbb{R}} \|\mathfrak{p}\underline{\Psi}|_{\Sigma_t}\|_t \|\mathfrak{p}\underline{\widetilde{\Psi}}|_{\Sigma_t}\|_t dt.$$

As discussed in Lemma 3.1 in [FMR16a], we can control the latter integrands with

$$\|\mathfrak{p}\underline{\widetilde{\Psi}}|_{\Sigma_t}\|_t \leq \sqrt{|I|}\,\|\underline{\widetilde{\Psi}}\|_{\oplus} = \sqrt{|I|}\,\|\underline{\widetilde{\psi}}\|_{\oplus}, \qquad \|\mathfrak{p}\underline{\Psi}|_{\Sigma_t}\|_t \leq \frac{C(\underline{\Psi})}{1+t^2},$$

where the constant $C(\underline{\Psi})$ depends on the spatial Sobolev norm of $\underline{\Psi}|_{\Sigma_t}$. We can conclude that

$$|N_{\mathscr{R}}(\underline{\psi}, \underline{\widetilde{\psi}})| \leq c(\underline{\Psi})\|\underline{\widetilde{\psi}}\|_{\oplus},$$

thus (4.3) holds true. On account of Corollary 4.2.4 the mWMOP follows immediately. ■

CHAPTER 5

HADAMARD STATES ARISING FROM A DEFORMATION ARGUMENT

As the last result of this thesis, we construct a bridge between the free field theory of the massive Dirac fields and the massless one. Using the Møller-Dappiaggi operator, we first realise an isomorphism between the spaces of classical observables and later we extend this isomorphism to the algebraic level. At the end of the chapter, we introduce a deformation argument in the mass parameter space for the quasifree state on such algebras. Loosely speaking, such argument guarantees that, if we can construct a Hadamard state for a free field theory with a given value of the mass, then one can induce a counterpart for the massless case, fulfilling, also, the Hadamard condition.

5.1 An Isomorphism between Spaces of Classical Observables

From Section 2.2, we know that the kinematic configurations of the Dirac field are either spinors $\psi \in \Gamma(SM)$ or cospinors $\phi \in \Gamma(S^*M)$. On these configurations, we define the following class of functionals:

$$
\begin{aligned}
S_\tau : \Gamma(SM) \to \mathbb{C}, \qquad & \psi \mapsto \langle \psi \mid \tau \rangle_{(s)} \\
C_\varsigma : \Gamma(S^*M) \to \mathbb{C}, \qquad & \phi \mapsto \langle \phi \mid \varsigma \rangle_{(c)}
\end{aligned}
\tag{5.1}
$$

where $\langle \cdot \mid \cdot \rangle_{(c)} = \langle A^{-1} \cdot \mid A^{-1} \cdot \rangle$, A is the adjunction map 2.5 and $\langle \cdot \mid \cdot \rangle_{(s)} = \langle \cdot \mid \cdot \rangle$ is the usual space-time inner product (2.8). Since both $\langle \cdot \mid \cdot \rangle_{(s)}$ and $\langle \cdot \mid \cdot \rangle_{(c)}$ induce non-degenerate bilinear pairings on $\Gamma_c(SM) \times \Gamma(SM)$ and on $\Gamma_c(S^*M) \times \Gamma(S^*M)$ respectively, we can identify the vector spaces $\{S_\tau \mid \tau \in \Gamma_c(SM)\}$ and $\{C_\varsigma \mid \varsigma \in \Gamma_c(S^*M)\}$ with $\Gamma_c(SM)$ and $\Gamma_c(S^*M)$ using the anti-linear maps

$$
\Gamma_c(SM) \ni \tau \mapsto S_\tau \qquad \text{and} \qquad \Gamma_c(S^*M) \ni \varsigma \mapsto C_\varsigma .
$$

At this stage, the dynamics is not yet implemented. To this end, we have to restrict the functionals 5.1 on $\mathrm{Sol}(\mathscr{D}_m)$ and $\mathrm{Sol}(\mathscr{D}_m^*)$, introduced in Section 2.3. With an abuse of notation, let us denote this restriction by the same symbols

$$
S_\tau : \mathrm{Sol}(\mathscr{D}_m) \to \mathbb{C} \qquad \text{and} \qquad C_\varsigma : \mathrm{Sol}(\mathscr{D}_m^*) \to \mathbb{C} .
$$

These functionals are not represented faithfully by $\Gamma_c(SM)$ and $\Gamma_c(S^*M)$ after the restriction to dynamical configurations. To get rid off these redundancies, we quotient out the functionals which vanish on the dynamical configuration:

$$N_{(s)} := \{\tau_m \in \Gamma_c(SM) \,|\, S_{\tau_m}(\psi_m) = 0\ \forall \psi_m \in \mathrm{Sol}(\mathscr{D}_m)\} \equiv \mathscr{D}_m\big(\Gamma_c(SM)\big)$$
$$N_{(c)} := \{\varsigma_m \in \Gamma_c(S^*M) \,|\, C_{\varsigma_m}(\phi_m) = 0\ \forall \phi_m \in \mathrm{Sol}(\mathscr{D}^*_m)\} \equiv \mathscr{D}^*_m\big(\Gamma_c(S^*M)\big).$$

Definition 5.1.1. *We call **spaces of classical observables** respectively for spinors and for cospinors*

$$\mathscr{E}^{(s)}_m := \left\{ S_{[\tau_m]} \,\middle|\, \forall \psi_m \in \mathrm{Sol}(\mathscr{D}_m)\,, \forall [\tau_m] \in \Gamma_c(SM)/\mathscr{D}_m\big(\Gamma_c(SM)\big) \text{ then } S_{[\tau_m]} := \langle \psi_m \mid \tau_m \rangle_{(s)} \right\},$$
$$\mathscr{E}^{(c)}_m := \left\{ C_{[\varsigma_m]} \,\middle|\, \forall \phi_m \in \mathrm{Sol}(\mathscr{D}^*_m)\,, \forall [\varsigma_m] \in \Gamma_c(S^*M)/\mathscr{D}^*_m\big(\Gamma_c(S^*M)\big) \text{ then } C_{[\varsigma_m]} := \langle \phi_m \mid \varsigma_m \rangle_{(c)} \right\}.$$

Using the Møller-Dappiaggi operator, in Section 4.2 we realised an isomorphism between spaces of solutions with different mass. Our goal is to implement an isomorphism between the spaces of observables for massive and massless Dirac fields. To achieve this goal, first we need to define the formal dual of Møller-Dappiaggi operator for the (co)spinor fields.

Remark 5.1.1. *Notice that, whenever the subscript m is missing, we refer to the massless case.*

To achieve this goal, let us first introduce the Møller-Dappiaggi operator for cospinor fields. Let $\Sigma_\pm$ be two Cauchy surfaces such that Σ_+ lies in the future of Σ_-. Let $\rho^\pm \in C^\infty(\mathbb{R})$ be a non decreasing function such that $\rho^+|_{J^+(\Sigma_+)} = 1$, $\rho^+|_{J^-(\Sigma_-)} = 0$ and $\rho^- = 1 - \rho^+$. Now let us define $m^\pm := m\rho^\pm$. For any $\phi_m \in \mathrm{Sol}(\mathscr{D}^*)$, we define the Møller-Dappiaggi operator for cospinor fields as

$$R_{(c)} = \left(\mathrm{Id} - E^{*-} m^-\right) \circ \left(\mathrm{Id} + E^{*+}_{m^+} m^+\right) : \mathrm{Sol}(\mathscr{D}^*_m) \to \mathrm{Sol}(\mathscr{D}^*)$$

being $E^{*+}_{m^+}$ the advanced Green operator for the $\mathscr{D}^*_{m^+}$, $E^{*-}_{m^-}$ and the retarded one for $\mathscr{D}^*_{m^-}$. The last ingredient needed in order to define the formal dual of $R_{(c)}$ is the pairing

$$\widetilde{\langle\,|\,\rangle}_{(c)} : \mathrm{Sol}(\mathscr{D}^*) \times \frac{\Gamma_c(S^*M)}{\mathscr{D}^*\big(\Gamma_c(S^*M)\big)} \to \mathbb{C}, \qquad \widetilde{\langle \phi \mid [\varsigma] \rangle}_{(c)} := \langle \phi \mid \varsigma \rangle_{(c)} = \int_M \prec \phi \mid \varsigma \succ_{(c)} d\mu_g$$

where $\prec \phi \mid \varsigma \succ_{(c)} = \varsigma\big(A^{-1}\phi\big)$.

Lemma 5.1.1. *Let $R^*_{(c)} : \mathscr{E}^{(c)} \to \mathscr{E}^{(c)}_m$ be such that, for every $C_{[\varsigma]} \in \mathscr{E}^{(c)}$,*

$$R^*_{(c)}[\varsigma] := \left[\left(\mathrm{Id} - m^- E^{*+}\right) \circ \left(\mathrm{Id} + m^+ E^{*-}_{m^+}\right)\varsigma\right],$$

where ς is any representative of $[\varsigma]$. Then $R^*_{(c)}$

- *is the formal dual operator to $R_{(c)}$ with respect to $\widetilde{\langle\,|\,\rangle}_{(c)}$;*

- *realizes an isomorphism of vector spaces between $\mathscr{E}^{(c)}$ and $\mathscr{E}^{(c)}_m$.*

Proof. Let $\phi_m \in \mathrm{Sol}(\mathscr{D}^*_m)$ and $[\varsigma] \in \Gamma_c(S^*M)/\mathscr{D}^*\big(\Gamma_c(S^*M)\big)$ be arbitrary. It holds

$$\widetilde{\langle R_{(c)}\phi_m \mid [\varsigma] \rangle}_{(c)} = \langle \phi_m \mid \varsigma \rangle_{(c)} + \langle E^{*+}_{m^+} m^+ \phi_m \mid \varsigma \rangle_{(c)} - \langle E^{*-} m^- \phi_m \mid \varsigma \rangle_{(c)} - \langle E^{*-} m^- E^{*+}_{m^+} m^+ \phi_m \mid \varsigma \rangle_{(c)}$$

Being $\mathscr{D}^*_{m^-}$ formally self-adjoint with respect to $\langle\,|\,\rangle_{(c)}$, we can rewrite the second term on the right hand as

$$\begin{aligned}\langle E^{*+}_{m^+} m^+ \phi_m \mid \varsigma \rangle_{(c)} &= \int_M \prec E^{*+}_{m^+} m^+ \phi_m \mid \varsigma \succ_{(c)} d\mu_g = \int_M \prec E^{*+}_{m^+} m^+ \phi_m \mid \mathscr{D}^*_{m^+} E^{*-}_{m^+} \varsigma \succ_{(c)} d\mu_g = \\ &= \int_M \prec m^+ \phi_m \mid E^{*-}_{m^+} \varsigma \succ_{(c)} d\mu_g = \int_M \prec \phi_m \mid m^+ E^{*-}_{m^+} \varsigma \succ_{(c)} d\mu_g = \langle \phi_m \mid m^+ E^{*-}_{m^+} \varsigma \rangle_{(c)} ,\end{aligned}$$

where we used that $\mathscr{D}^*_{m^+} E^{*-}_{m^+} = \mathrm{Id}_{\Gamma_c(S^*M)}$ and that $\mathrm{supp}(m^+u) \cap \mathrm{supp}(E^{*-}_{m^+}\varsigma)$ is compact. A similar expression is obtained also for the other terms

$$\begin{aligned}\langle E^{*-} m^- \phi_m \mid \varsigma \rangle_{(c)} &= \int_M \prec E^{*-} m^- \phi_m \mid \varsigma \succ_{(c)} d\mu_g = \\ &= \int_M \prec \phi_m \mid m^- E^{*+} \varsigma \succ_{(c)} d\mu_g = \langle \phi_m \mid m^- E^{*+} \varsigma \rangle_{(c)} ; \\ \langle E^{*-} m^- E^{*+}_{m^+} m^+ \phi_m \mid \varsigma \rangle_{(c)} &= \int_M \prec E^{*+}_{m^+} m^+ E^{*-} m^- \phi_m \mid \varsigma \succ_{(c)} d\mu_g = \\ &= \int_M \prec \phi_m \mid m^- E^{*+} m^+ E^{*-}_{m^+} \varsigma \succ_{(c)} d\mu_g = \langle \phi_m \mid m^- E^{*+} m^+ E^{*-}_{m^+} \varsigma \rangle_{(c)} .\end{aligned}$$

Since, for all $\mathscr{D}^* f$, we have

$$\int_M \prec R_{(c)} \psi_m \mid \mathscr{D}^* f \succ_{(c)} d\mu_g = \int_M \prec \mathscr{D}^* \circ R_{(c)} \psi_m \mid f \succ_{(c)} d\mu_g = 0 ,$$

with $f \in \Gamma_c(S^*M)$, $\psi_m \in \mathrm{Sol}(\mathscr{D}^*_m)$ and $R_{(c)}\psi_m \in \mathrm{Sol}(\mathscr{D}^*)$, we can conclude that all the equalities obtained do not depend on the choice of representative. Merging all together, we obtain:

$$\langle \widetilde{R_{(c)}\phi_m \mid [\varsigma]} \rangle_{(c)} = \left\langle \phi_m \mid \left(\mathrm{Id} - m^- E^{*+}\right) \circ \left(\mathrm{Id} + m^+ E^{*-}_{m^+}\right) \varsigma \right\rangle_{(c)} = \left\langle \widetilde{\phi_m \mid R^*_{(c)}[\varsigma]} \right\rangle_{(c)}$$

Moreover $R^*_{(c)}$ is an isomorphism of vector spaces, since the operator

$$R^{*-1}_{(c)} := \left(\mathrm{Id} - m^+ E^{*-}\right) \circ \left(\mathrm{Id} + m^- E^{*+}_{m^-}\right)$$

is the inverse operator:

$$\begin{aligned}&R^*_{(c)} R^{*-1}_{(c)} = \\ &= \left(\mathrm{Id} - m^- E^{*+}\right) \circ \left(\mathrm{Id} + m^+ E^{*-}_{m^+}\right) \circ \left(\mathrm{Id} - m^+ E^{*-}\right) \circ \left(\mathrm{Id} + m^- E^{*+}_{m^-}\right) = \\ &= \left(\mathrm{Id} - m^- E^{*+}\right) \circ \left(\mathrm{Id} + m^+ E^{*-}_{m^+} - m^+ E^{*-} - m^+ E^{*-}_{m^+} m^+ E^{*-}\right) \circ \left(\mathrm{Id} + m^- E^{*+}_{m^-}\right) = \\ &= \left(\mathrm{Id} - m^- E^{*+}\right) \circ \left(\mathrm{Id} + m^+ E^{*-}_{m^+} - m^+ E^{*-} - m^+ E^{*-}_{m^+} (-\mathscr{D}_{m^+} + \mathscr{D}) E^{*-}\right) \circ \left(\mathrm{Id} + m^- E^{*+}_{m^-}\right) = \\ &= \left(\mathrm{Id} - m^- E^{*+}\right) \circ \left(\mathrm{Id} + m^+ E^{*-}_{m^+} - m^+ E^{*-} + m^+ E^{*-} - m^+ E^{*-}_{m^+}\right) \circ \left(\mathrm{Id} + m^- E^{*+}_{m^-}\right) = \\ &= \left(\mathrm{Id} - m^- E^{*+}\right) \circ \left(\mathrm{Id} + m^- E^{*+}_{m^-}\right) = \\ &= \mathrm{Id} - m^- E^{*+} + m^- E^{*+}_{m^-} - m^- E^{*+} m^- E^{*+}_{m^-} = \\ &= \mathrm{Id} - m^- E^{*+} + m^- E^{*+}_{m^-} - m^- E^{*+} (\mathscr{D} - \mathscr{D}_{m^-}) E^{*+}_{m^-} = \\ &= \mathrm{Id} - m^- E^{*+} + m^- E^{*+}_{m^-} - m^- E^{*+}_{m^-} + m^- E^{*+} = \mathrm{Id}\end{aligned}$$

Analogously one shows that $R^{*-1}_{(c)} R^*_{(c)}$ =Id and this concludes the proof. ■

This analysis is extended straightforwardly to the spinor fields. As before we define the pairing $\widetilde{\langle \mid \rangle}_{(s)}$ between $\mathrm{Sol}(\mathscr{D})$ and $\Gamma_c(S^M)/\mathscr{D}\big(\Gamma_c(S^M)\big)$ as

$$\widetilde{\langle \psi \mid [\tau] \rangle}_{(s)} := \langle \psi \mid \tau \rangle_{(s)} = \int_M \prec \psi \mid \tau \succ d\mu_g \,.$$

The formal dual operator to $R_{(s)}$ with respect to $\widetilde{\langle \mid \rangle}_{(s)}$ for spinor fields is

$$R^*_{(s)} := \Big(\mathrm{Id} - m^- E^+\Big) \circ \Big(\mathrm{Id} + m^+ E^-_{m^+}\Big) : \mathscr{E}^{(s)} \to \mathscr{E}^{(s)}_m \,,$$

where τ is any representative of $[\tau]$. $R^*_{(s)}$ realises an isomorphism of vector spaces between $\mathscr{E}^{(s)}$ and $\mathscr{E}^{(s)}_m$.

5.2 The Deformation Argument

In the previous section, we saw that the dual Møller-Dappiaggi operator induces an isomorphism between the space of classical observables for massive and massless Dirac fields. These spaces are the building blocks of the algebra of Dirac fields once a Hermitian structure is introduced. Let us recall the construction of algebra of Dirac fields using $\mathscr{E}^{(s)}$ and $\mathscr{E}^{(c)}$. First of all, we endow such spaces with the Hermitian forms:

$$h^s : \mathscr{E}^{(s)} \times \mathscr{E}^{(s)} :\to \mathbb{C}\,, \qquad h^s(S_{[\tau]}, S_{[\widetilde{\tau}]}) := i\,\langle \tau \mid E\widetilde{\tau} \rangle_{(s)} \tag{5.2}$$

$$h^c : \mathscr{E}^{(c)} \times \mathscr{E}^{(c)} :\to \mathbb{C}\,, \qquad h^c(C_{[\varsigma]}, C_{[\widetilde{\varsigma}]}) := -i\,\big\langle \varsigma \mid E^* \widetilde{\varsigma} \big\rangle_{(c)} \tag{5.3}$$

being E and E^* the causal propagator associated to $\mathscr{D}$ and $\mathscr{D}^*$ respectively and where the representatives $\varsigma \in [\varsigma]$, $\widetilde{\varsigma} \in [\widetilde{\varsigma}]$, $\tau \in [\tau]$ and $\widetilde{\tau} \in [\widetilde{\tau}]$ are chosen arbitrarily. Using the adjunction map A, we can define the algebra of Dirac fields as:

$$\mathfrak{F}(M) := \frac{T(\mathscr{E}^{(s)} \oplus \mathscr{E}^{(c)})}{\mathfrak{I}}$$

where $T(\mathscr{E}^{(s)} \oplus \mathscr{E}^{(c)})$ is the tensor algebra and $\mathfrak{I}$ is the closed $*$-ideal generator by the element

$$\mathbf{1}_{\mathfrak{F}} = \{1, 0, \dots\}, \qquad \Psi(S_{[\tau]}) = \left\{0, \begin{pmatrix} S_{[\tau]} \\ 0 \end{pmatrix}, 0, \dots \right\} \qquad \text{and} \qquad \Phi(C_{[\varsigma]}) = \left\{0, \begin{pmatrix} 0 \\ C_{[\varsigma]} \end{pmatrix}, 0, \dots \right\}$$

satisfying for any $S_{[\tau]}, S_{[\widetilde{\tau}]} \in \mathscr{E}^{(s)}$ and any $C_{[\varsigma]}, C_{[\widetilde{\varsigma}]} \in \mathscr{E}^{(c)}$ the following relations

(i) $\Psi(S_{[\tau]})^* = \Phi(C_{[A\tau]})$;

(ii) $\big\{\Psi(S_{[\tau]}), \Psi(S_{[\widetilde{\tau}]})\big\} = 0 = \big\{\Phi(C_{[\varsigma]}), \Phi(C_{[\widetilde{\varsigma}]})\big\}$;

(iii) $\big\{\Psi(S_{[\tau]}), \Phi(C_{[\varsigma]})\big\} = h^s\big(S_{[\tau]}, S_{A^{-1}[\varsigma]}\big)\, \mathbf{1}_{\mathfrak{F}}$.

Lemma 5.2.1. *The operators $R^*_{(s)}$ and $R^*_{(c)}$ preserve the Hermitian forms* (5.2) *and* (5.3).

Proof. Let $S_{[\tau]}, S_{[\tilde{\tau}]} \in \mathscr{E}^{(s)}$ and choose two representatives $\tau, \tilde{\tau} \in \Gamma_c(SM)$. Then we have:

$$h^s(S_{R^*_{(s)}[\tau]}, S_{R^*_{(s)}[\tilde{\tau}]}) = i\left\langle R^*_{(s)}\tau \mid E_m R^*_{(s)}\tilde{\tau}\right\rangle_{(s)} = i\left\langle \tau \mid R_{(s)} E_m R^*_{(s)}\tilde{\tau}\right\rangle_{(s)}.$$

Since $R_{(s)}$ has an inverse, let us denote $r_s := \mathrm{Id} - R^{-1}_{(s)}$. By direct computation we obtain $R_{(s)} - \mathrm{Id} = R_{(s)} r_s = r_s R_{(s)}$. Using the previous relation and its dual, $R_{(s)} E_m R^*_{(s)}$ be can factorised as

$$R_{(s)} E_m R^*_{(s)} = R_{(s)} E^+_m - E^-_m R^*_{(s)} + R_{(c)} E^+_m r^*_s R^*_{(s)} - R_{(s)} r_s E^-_m R^*_{(s)}.$$

Notice that the last two summands cancel each other because $E^+_m r^*_s = r_s E^-_m$. Using $R_{(s)} E^+_m = E^+$ and $E^-_m R^*_{(s)} = E^-$, we can conclude. ■

Thanks to Lemma 5.2.1, it turns out that also the ideal $\mathfrak{I}$ is isomorphic to $\mathfrak{I}_m$. Indicating the extension of $R^*_{(s)} \oplus R^*_{(c)}$ to $\mathfrak{F}(M)$ as R^* we have identified the isomorphism

$$R^* : \mathfrak{F}(M) \to \mathfrak{F}_m(M), \tag{5.4}$$

where the action is unambiguously defined on the generators by $R^*_{(s)} \oplus R^*_{(c)}$.

As the last result of this Chapter (and of this thesis), we address the question asked in the Introduction: *Given a Hadamard state for a free field theory with a given value of the mass, can we build a counterpart for the massless Dirac field such that the same property holds?*

Theorem 5.2.1. *Let M be a globally hyperbolic spacetime. Let $\mathfrak{F}_m(M)$ be the associated algebra of (massive) Dirac fields and let $\omega_m : \mathfrak{F}_m(M) \to$ C be a quasifree Hadamard state. Moreover, let $R^* : \mathfrak{F}(M) \to \mathfrak{F}_m(M)$ be the isomorphism* (5.4). *Then*

$$\omega := \omega_m \circ R^* : \mathfrak{F}(M) \to \mathbb{C} \qquad a \mapsto \omega(a) := \omega_m(R^* a)$$

is a quasifree Hadamard state.

Proof. Since ω is defined composing ω with the isomorphism R^*, it inherits the property of being a quasifree state. In order to check whether ω satisfies the Hadamard condition, let us consider its two-point function:

$$\begin{aligned}\omega\big(S_{[\tau]} \oplus C_{[\varsigma]} \otimes S_{[\tilde{\tau}]} \oplus C_{[\tilde{\varsigma}]}\big) &= \omega_m\big(S_{R^*_{(s)}[\tau]} \oplus C_{R^*_{(c)}[\varsigma]} \otimes S_{R^*_{(s)}[\tilde{\tau}]} \oplus C_{R^*_{(c)}[\tilde{\varsigma}]}\big) = \\ &= \omega_{2,m}\big(R^*_{(s)}(\tau) \oplus R^*_{(c)}(\varsigma), R^*_{(s)}(\tilde{\tau}) \oplus R^*_{(c)}(\tilde{\varsigma})\big)\end{aligned}$$

for any $S_{[\tau]} \oplus C_{[\varsigma]}$, $S_{[\tilde{\tau}]} \oplus C_{[\tilde{\varsigma}]} \in \mathscr{E}^{(s)} \oplus \mathscr{E}^{(c)}$. Here $\omega_{2,m} \in \left(\Gamma_c\big(SM \oplus S^*M\big)^{\otimes 2}\right)'$ is the bi-distribution associated to ω_m. Consider now the restriction of $\omega_{2,m}$ to a neighbourhood $\mathscr{O}$ of a Cauchy surface $\tilde{\Sigma} \subset J^+(\Sigma^+)$. This implies that $\rho^- = 0$ and then $\omega_{2,m}$ reads

$$\omega_{2,m}|_{\mathscr{O}} = \omega_{2,m}\left(\left(\mathrm{Id} + m^+ E^-_{m^+}\right)(\bullet) \oplus \left(\mathrm{Id} + m^+ E^{*-}_{m^+}\right)(\bullet), \left(\mathrm{Id} + m^+ E^-_{m^+}\right)(\bullet) \oplus \left(\mathrm{Id} + m^+ E^{*-}_{m^+}\right)(\bullet)\right) := \tilde{\omega}_{2,m}.$$

Then $\omega_{2,m}|_{\mathscr{O}}$ has the same singularities of $\omega^+_{2,m}$, and, applying the theorem of propagation of the singularities [DH72], it has the same singularities in the whole manifold. Since these two states have the same singularities, we consider the latter one and we restrict it to a neighbourhood $\mathscr{O}'$ of a Cauchy surface $\check{\Sigma} \subset J^-(\Sigma^-)$. This implies that $\rho^+ = 0$ and

$$\widetilde{\omega}_{2,m}\big|_{\mathscr{O}'} = \omega_{2,m}\big((\cdot) \oplus (\cdot), (\cdot) \oplus (\cdot)\big).$$

Since $\omega_{2,m}$ is Hadamard per hypothesis, applying once again the theorem of propagation of singularities, we obtain that also $\widetilde{\omega}_{2,m}$ is of Hadamard form. Hence, we can conclude that ω_2 is a Hadamard state. ■

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