Looking at the quantum states with the eyes of algebraic quantum field theory

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Motivations

" *What is a* **QFT?** " $\xrightarrow{\text{for a deeper understanding}}$ **mathematical axioms** for **QFT**

Locally covariant QFT

✓ $\mathfrak{A} : \mathsf{Loc} \to \mathsf{Alg}$

- **Locality** if $f : \mathcal{M}_1 \hookrightarrow \mathcal{M}_2 \implies \mathfrak{f} : \mathcal{A}(\mathcal{M}_1) \to \mathcal{A}(\mathcal{M}_2)$ injective
- **Causality:** if $\mathcal{M}_1 \xhookrightarrow{f_1} \mathcal{M} \xhookleftarrow{f_2} \mathcal{M}_2$ caus. disjoint $\implies [\mathfrak{f}_1\mathcal{A}(\mathcal{M}_1), \mathfrak{f}_2\mathcal{A}(\mathcal{M}_2)] = 0$
- **Time-Slice axiom:** if $f : \mathcal{M}_1 \to \mathcal{M}_2$ s.t. $f(\mathcal{M}_1) \supset \Sigma_2 \implies \mathfrak{f}$ is isomorphism

✓ $\omega : \mathcal{A}(\mathcal{M}) \to \mathbb{C}$ s.t. positive $\omega(a^*a) \geq 0$ and normalized $\omega(1_{\mathcal{A}}) = 1$

¿ **natural state** ? $\mathcal{M}_1 \xhookleftarrow{f_1} \mathcal{M} \xhookrightarrow{f_2} \mathcal{M}_2 \Rightarrow \omega_{\mathcal{M}_2} \circ \mathfrak{f}_2 = \omega_{\mathcal{M}} = \omega_{\mathcal{M}_1} \circ \mathfrak{f}_1$

↯ scalarQFT: C. J. Fewster and R. Verch - Annales Henri Poincare 13 (2012)

Goals:

1) investigate 'natural state' in Topological QFT

2) realize a suitable construction for Dirac fields

Outline of the Talk

- **Natural states for Abelian Chern-Simons theory**
- **Algebraic approach to quantum Dirac fields**
- **Quantum states for Rindler spacetimes**

Based on:

- C. Dappiaggi, S.M., A. Schenkel - J. Geom. Phys. **116** (2017)
- F. Finster, S.M., C. Röken - to appear on J. Math. Anal. Appl. (2017)
- My Ph.D. Thesis

PART I:

Natural States for Chern-Simons Theory

Classical observables for Abelian Chern-Simons theory

[C. Dappiaggi, S. M. and A. Schenkel: ... *natural states for Abelian Chern-Simons theory* - Journal of Geometry and Physics **116** (2017).]

- We consider $\mathcal{M} \simeq \mathbb{R} \times \Sigma$ oriented and with $\dim \Sigma = 2$
- The action of **Abelian Chern–Simons theory**

$$S = \frac{1}{4\pi} \int_M A \wedge dA \quad \Rightarrow \quad 0 = \frac{\delta S}{\delta A} = \frac{1}{2\pi} dA$$

- The moduli space of flat $U(1)$-connection:

$$\mathrm{Flat}_{U(1)} := \frac{\Omega^1_d(M)}{\Omega^1_{\mathbb{Z}}(M)} \simeq \frac{H^1(M;\mathbb{R})}{H^1(M;\mathbb{Z})} \simeq \frac{H^1(\Sigma;\mathbb{R})}{H^1(\Sigma;\mathbb{Z})} \simeq \frac{\Omega^1_d(\Sigma)}{\Omega^1_{\mathbb{Z}}(\Sigma)}$$

- As **classical osservables** we take all group characters $:= \mathrm{Hom}\big(\mathrm{Flat}_{U(1)}(\Sigma), U(1)\big)$:

$$\text{given any } \varphi \in \Omega^1_c(\Sigma) \qquad A \mapsto \exp\left(2\pi\imath \int_\Sigma \varphi \wedge A\right)$$

- This character descends to the quotient if and only if

$$\text{group character} \simeq H^1_c(\Sigma;\mathbb{Z}) := \left\{[\varphi] \in H^1_c(\Sigma;\mathbb{R}) : \int_\Sigma \varphi \wedge H^1(\Sigma) \subseteq \mathbb{Z}\right\} \simeq \left(\mathbb{Z}^{N(g,p)}\right)$$

Quantization of Abelian Chern-Simons theory

[C. Dappiaggi, S. M. and A. Schenkel: ... *natural states for Abelian Chern-Simons theory* - Journal of Geometry and Physics **116** (2017).]

- We construct a $*$-algebra $\Delta :=\text{span}\,\{W_{[\varphi]}\,|\,[\varphi]\in H^1_c(\Sigma;\mathbb{Z})\}$ where $W_{[\,.\,]}$ satisfy

$$W_{[\varphi]}\,W_{[\widetilde{\varphi}]} := e^{-i\hbar\,\tau_\Sigma([\varphi],[\widetilde{\varphi}])}\;W_{[\varphi]+[\widetilde{\varphi}]}\,, \quad {W_{[\varphi]}}^* := W_{-[\varphi]}\,, \quad \tau([\varphi],[\widetilde{\varphi}])_\Sigma = \int_\Sigma \varphi\wedge\widetilde{\varphi}$$

- We obtain a **C*-algebra** taking the completion of Δ with respect to the norm

$$||a||^{m.r.n} := \sup_{\omega\in\mathcal{F}} \sqrt{\omega(a^*a)}$$

where $\omega:\Delta\to\mathbb{C}$ is a state, namely $\omega(1_\Delta)=1$ and $\omega(a^*a)\geq 0$.

- An **invariant functional** under the action of the symplectic group

$$\omega\big(W_{T[\varphi]}\big) = \omega\big(W_{[\varphi]}\big) := \begin{cases} 1 & \text{if } [\varphi]=0 \\ K_{[\varphi]} & \text{else} \end{cases}$$

Q: How many invariant states exist on this C^*-algebra?

↯ $\mathrm{Sp}(\mathbb{Z}^{N(g,p)},\tau_\Sigma)\not\subset \mathrm{O}(\mathbb{Z}^{N(g,p)},\mu)$, *does not exist* an invariant Gaussian state

$$\omega\big(W_{[\varphi]}\big) = e^{-\mu([\varphi],[\varphi])}$$

Non-existence of natural states

[C. Dappiaggi, S. M. and A. Schenkel: *Non-existence of natural states for Abelian Chern-Simons theory* - Journal of Geometry and Physics **116** (2017).]

Theorem

There exists *no natural state* for the functor $\mathcal{A} : \mathsf{Man}_2 \to \mathsf{CAlg}$, namely a state for each Σ such that for all Man_2-morphisms $f : \Sigma \to \Sigma'$ holds true:

$$\omega_{\Sigma'} \circ \mathcal{A}(f) = \omega_\Sigma$$

Sketch of the proof

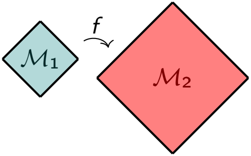
- Let us assume that there exists a natural state $\{\omega_\Sigma\}_{\Sigma \in \mathsf{Man}}$
- Consider the Man_2-diagram: $\mathbb{S}^2 \xleftarrow{f_1} \mathbb{R} \times \mathbb{S} \xrightarrow{f_2} \mathbb{T}^2$
- The naturality of the state implies: $\omega_{\mathbb{S}^2} \circ \mathcal{A}(f_1) = \omega_{\mathbb{R}\times\mathbb{S}} = \omega_{\mathbb{T}^2} \circ \mathcal{A}(f_2)$
- Because of $H^1_c(\mathbb{S}^2; \mathbb{Z}) = 0$, then $\mathcal{A}(\mathbb{S}^2) \simeq \mathbb{C}$ and hence $\omega_{\mathbb{S}^2} = \mathrm{id}_\mathbb{C}$ is unique on $\mathbb{C}$
- We can choose f_2 such that $W_n^{\mathbb{R}\times\mathbb{T}} \mapsto W_{(n,0)}^{\mathbb{T}^2}$ we obtain that $\omega_{\mathbb{T}^2}\big(W_{(n,0)}^{\mathbb{T}^2}\big) = 1$
- Choosing $a = \alpha_1 \, 1 + \alpha_2 \, W_{(1,1)}^{\mathbb{T}^2} + \alpha_3 \, W_{(0,1)}^{\mathbb{T}^2} \in \mathcal{A}(\mathbb{T}^2)$ the functional $\omega_{\mathbb{T}^2}(a^* a) < 0$

Q.E.D

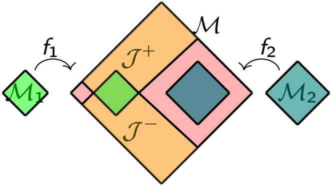
First conclusion and new goal!

Locally covariant QFT: Loc $\to$ Alg

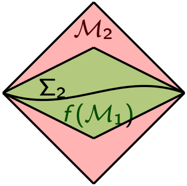
✓ **Locality**



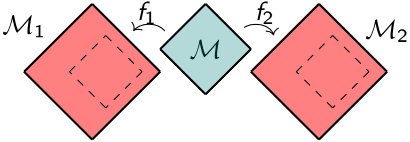
✓ **Causality**



✓ **Time-Slice axiom**



↯ **natural state** $\omega_{\mathcal{M}_2} \circ \mathfrak{f}_2 \neq \omega_{\mathcal{M}_1} \circ \mathfrak{f}_1$



GOAL: **construct ω non-locally determined by the geometry**

PART II:

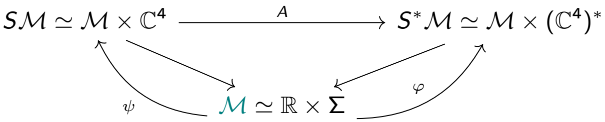
Quasifree States for CAR Algebras

Spin geometry on globally hyperbolic spacetimes

- $\mathcal{M} \simeq \mathbb{R} \times \Sigma$ is 4-dim **globally hyperbolic spacetime**

$$ds^2 = \beta^2 dt^2 - h_t; \quad \beta \in C^\infty(M; \mathbb{R}^+) \text{ and } h_t \in \text{Riem}(\Sigma); \forall t \in \mathbb{R}$$

- **Spinor bundle** $S\mathcal{M}$ and **cospinor bundle** $S^*\mathcal{M}$



- **Spin product** $\prec \cdot \mid \cdot \succ_x \colon \Gamma(S\mathcal{M}) \times \Gamma(S\mathcal{M}) \to \mathbb{C}$

$$\prec \psi \mid \widetilde{\psi} \succ_x := ((A\psi)\widetilde{\psi})(x).$$

- **Scalar products** for spinor $(\cdot \mid \cdot)^{(s)}$ and cospinor fields $(\cdot \mid \cdot)^{(c)}$

$$(\cdot \mid \cdot)^{(s)} \doteq \int_\Sigma \prec \cdot \mid \not{\psi} \cdot \succ_x \, d\Sigma \qquad (\cdot \mid \cdot)^{(c)} \doteq \int_\Sigma \prec A^{-1} \cdot \mid \not{\psi} A^{-1} \cdot \succ_x \, d\Sigma$$

$*$-algebras and quasifree states

[H. Araki: *On quasifree states of CAR and Bogoliubov automorphisms* - Publ. Res. Inst. Math. Sci. Kyoto 6 (1971).]

- INGREDIENTS:
  - Hilbert space $\mathscr{H}$
  - Anti-unitary involution $\Upsilon : \mathscr{H} \to \mathscr{H}$ • $\Upsilon^2 = Id$ • $(\Upsilon f \mid \Upsilon g) = (g \mid f)$

Definition

- A **CAR algebra** $\mathcal{A}$ over $(\mathscr{H}, \Upsilon)$ is a $*$-algebra generated by $B(f)$, $B(f)^*$ and $1_{\mathcal{A}}$:
  1) $B(f)$ is (complex) linear in f
  2) $B(f)^* = B(\Upsilon f)$
  3) $B(f)B(g)^* + B(g)^* B(f) = (f \mid g)\, 1_{\mathcal{A}}$
- **Quasifree state** $\omega : \mathcal{A} \to \mathbb{C}$ linear

$$\bullet\ \omega\big(B(f)^* B(f)\big) \geq 0 \qquad \bullet\ \omega(1_{\mathcal{A}}) = 1 \qquad \bullet\ \omega_{2n+1}\big((B(f_1) \cdots B(f_{2n+1})\big) = 0$$

$$\bullet\ \omega_{2n}\big((B(f_1) \cdots B(f_{2n})\big) = \sum_{\sigma \in S'_{2n}} (-1)^{\mathrm{sign}(\sigma)} \prod_{i=1}^{n} \omega_2\big(B(f_{\sigma(2i-1)}) B(f_{\sigma(2i)})\big)$$

Characterization of quasifree states

Lemma *(H. Araki: On quasifree states of CAR and Bogoliubov automorphisms.)*

Let Υ be an involution on $\mathscr{H}$ and Q be a **bounded, symmetric operator** on $\mathscr{H}$

$$\text{(a) } Q + \Upsilon Q \Upsilon = Id \qquad \text{(b) } 0 \leq Q = Q^* \leq 1$$

Then there exists a **unique quasifree state** ω on $\mathcal{A}$ such that

$$\omega_2\big(B(f^*)B(g)\big) = (f \mid Qg)$$

Proposition 1 *(N. Drago and S. M.: arXiv:1607.02909)*

- 4dim-Globally hyperbolic spacetime $\mathcal{M} = \mathbb{R} \times \Sigma$ and (co)spinor $S^{(*)}\mathcal{M} = \mathcal{M} \times \mathbb{C}^4$
- Hilbert spaces $\mathscr{H}^s = \overline{\left(C^\infty_{sc}(\mathcal{M}, S\mathcal{M}), (\ \mid\ )_\Sigma\right)}$ and $\mathscr{H}^c = \overline{\left(C^\infty_{sc}(\mathcal{M}, S^*\mathcal{M}), (\ \mid\ )_\Sigma\right)}$
- Adjunction map $A : \mathscr{H}^s \to \mathscr{H}^c$
- Orthonormal projector Π on $\mathscr{H}^s$

Then setting $\mathscr{H} := \mathscr{H}^s \oplus \mathscr{H}^c \implies P := \Pi \oplus (Id - A\Pi A^{-1})$ satisfies (a) and (b)

PART III:

Quantum States in Rindler Spacetime

Quantum states in Rindler spacetime - I

[F. Finster, S. M. and C. Röken: *The fermionic signature operator and quantum states in Rindler space-time* - to appear on Journal of Mathematical Analysis and Applications (2017).]

$$\mathcal{R} = \left\{(t,x) \in \mathbb{R}^{1,1} \quad \text{with} \quad |t| < x\right\} \text{ with element line } ds^2 = dt^2 - dx^2$$

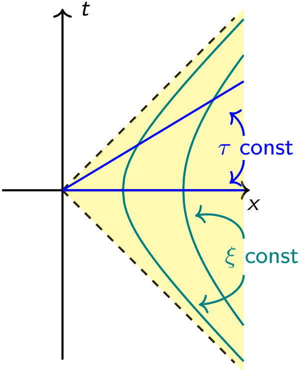
- Rindler coordinates:

$$\begin{cases} t = \xi \sinh \tau \\ x = \xi \cosh \tau \end{cases} \quad \Rightarrow \quad ds^2 = \xi^2 d\tau^2 - d\xi^2$$

- γ-matrices: $\{\gamma^a, \gamma^b\} = 2g^{ab}\, 1_{\mathbb{C}^2}$
- Dirac operator: $\mathcal{D} = i\gamma^0 \partial_t + i\gamma^1 \partial_x - 1_{\mathbb{C}^2} m$
- Solution space of Dirac equation

$$\mathcal{S}ol(\mathcal{D}) := \{\psi \in \Gamma_{sc}(S\mathcal{R}) \,|\, \mathcal{D}\psi = 0\}$$

- Hilbert space $\mathscr{H} := \overline{(\mathcal{S}ol(\mathcal{D}), (\cdot \mid \cdot))}$
- scalar product $(\cdot \mid \cdot) := \int_0^\infty \prec \cdot \mid \gamma^0 \cdot \succ \, dx$



Quantum states in Rindler spacetime - II

[F. Finster, S. M. and C. Röken: *The fermionic signature operator and quantum states in Rindler space-time* - to appear on Journal of Mathematical Analysis and Applications (2017).]

Theorem 1

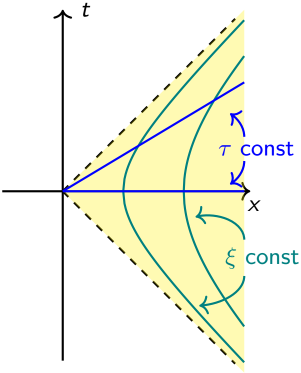
- $\forall \tilde{\psi} \in \mathscr{H}$, $\exists c = c(\tilde{\psi})$ s.t. for every $\psi \in \mathscr{H}$.

$$\left| \left\langle \psi \mid \tilde{\psi} \right\rangle \right| \doteq \left| \int_{\mathcal{R}} \prec \psi \mid \tilde{\psi} \succ dtdx \right| \leq c(\tilde{\psi}) \, \|\psi\|$$

- Fermionic signature operator $\mathcal{S} : D(\mathcal{S}) \subset \mathscr{H} \to \mathscr{H}$

$$\left\langle \psi \mid \tilde{\psi} \right\rangle = \left(\psi \mid \mathcal{S}\tilde{\psi} \right)$$

N.B.: $\mathcal{S}$ is densely defined, symmetric and unbounded.



- $\mathcal{S}$ has unique self adjoint extensions and $\sigma(\mathcal{S}) = \mathbb{R}$

- Fermionic projector $\Pi := \chi_{(0,+\infty)}(\mathcal{S}) \Longleftrightarrow$ quasifree state $\omega_{FP} : \mathcal{A} \to \mathbb{C}$

Quantum states in Rindler spacetime - III

[F. Finster, S. M. and C. Röken: *The fermionic signature operator and quantum states in Rindler space-time* - to appear on Journal of Mathematical Analysis and Applications (2017).]

- In the coordinates (τ, ξ), the Dirac equation takes the *Hamiltonian* form

$$i\partial_\tau \psi = \mathbf{H}\psi$$

Theorem 2

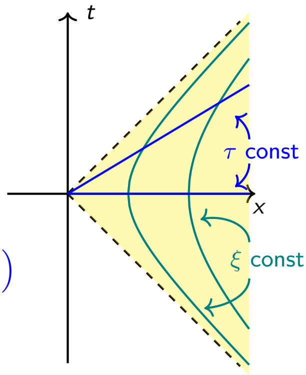
- In $\mathcal{R}^2 = \{(t,x) \in \mathbb{R}^{1,1} \quad \text{with} \quad |t| < x\}$
  - (i) fermionic signature operator: $\mathcal{S} = -\dfrac{\mathbf{H}}{\pi m}$
  - (ii) ground state: $\chi(\mathcal{S}) = \chi(\mathbf{H})$

- In $\mathcal{R}^4 = \{(t,x,y,z) \in \mathbb{R}^{1,3} \quad \text{with} \quad |t| < x\}$
  - (i) fermionic signature operator:

$$\mathcal{S} = -\frac{1}{\pi\sqrt{m^2 + k_y^2 + k_z^2}}\left(\mathrm{H} + \frac{1}{2m}\,\gamma^0\gamma^1\left(\gamma^2\partial_y + \gamma^3\partial_z\right)\right)$$

  - (ii) quasifree state: $\chi(\mathcal{S}) \neq \chi(\mathbf{H})$

    (¿physical interpretation?)



Are all states physically acceptable?

Of course not! Minimal requirements are:

i) covariant construction of Wick polynomials to deal with interactions,

ii) same UV behavior of the Minkowski vacuum,

iii) finite quantum fluctuations of all observables.

Answer: **Hadamard States**

- A (quasifree) state ω satisfies the **Hadamard condition** if and only if

$$WF(\omega_2) = \left\{ (x, y, \xi_x, \xi_y) \in T^*M^{\otimes 2} \setminus 0 \,\middle|\, (x, \xi_x) \sim (y, -\xi_y), \quad \xi_x \triangleright 0 \right\}$$

Question: **How many Hadamard states do we know?**

- deformation arguments (existence) - S. A. Fulling, F. J. Narcowich, R. M. Wald
- static or highly symmetric spacetimes - H. Sahlmann, R. Verch, K. Fredenhagen, ...
- Asymptotically flat spacetimes - C. Dappiaggi, V. Moretti , S.M., N. Pinamonti, ...
- Minkowski with external potential - F.Finster, S.M., C. Röken

Conclusions: THANK YOU for your attention!

What we "learn":

- The construction of a state should be determined non-locally by the geometry
- With every self-adjoint operator we can construct a quasifree state
- We could represent sesquilinear pairings to obtain self-adjoint operators

Benefit:

✓ The construction of the fermionic projector work without symmetries

Flaws:

↯ The construction of the fermionic projector works only for massive fields

Possible future investigation:

¿ Does the state in 4-dim Rindler satisfy the Hadamard condition ?

¿ Can we generalize this construction to integer spin particles?

¿ What do we obtain by applying this construction to time-depend models?