Hadamard States via Fermionic Projectors in a Time-Dependent External Potentials[1]

Simone Murro

Fakultät für Mathematik
Universität Regensburg

New Trends in Algebraic Quantum Field Theory

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[1] Joint work with F. Finster and C. Röken

Motivation

Our setting:

- *Space-time:* $\mathcal{M} = (\mathbb{R}^4, \eta)$, with $\dim\mathcal{M} = 4$ and $\eta = \text{diag}(+1, -1, -1, -1)$.
- *Matter:* $\big(\imath\gamma^\mu\partial_\mu + \mathcal{B} - m\big)\psi_m(x) = 0$.

Difficulties:

- No *translational symmetries* $\longrightarrow$ No canonical frequency splitting.
- No *conformal covariance* $\longrightarrow$ We cannot use the *"bulk to boundary corresponcence"*.

Goals:

- Construct a *Hadamard state*;
- Test our construction in an external time-dependent potential as preparation for curved space-times.

Warning

- Our construction is considerably **different** from the (modified) S-J construction:

  [Brum-Fredenhagen, arXiv:1307.0482 | Fewster-Verch, arXiv:1206.1562.]

- There exists **two** constructions of the fermionic projector:

  The *mass oscillation property* is something completely new!

Outline

- **Mathematical Preliminaries**
- **Quasi-Free States and Fermionic Projectors**
- **Mass Oscillation Property**
- **Hadamard States**

Based on:

- F. Finster, S. M., C. Röken, arXiv:1501.05522 [math-ph]

An Old Story I: Dirac spinor

- Spinors $\psi_m \in C^\infty_{sc}(\mathcal{M}, S\mathcal{M})$ and cospinors $\sigma \in C^\infty_{sc}(\mathcal{M}, S^*\mathcal{M})$.
- Dirac conjugation map $.^* \ : \ C^\infty_0(\mathcal{M}, S\mathcal{M}) \underset{(\leftarrow)}{\longrightarrow} C^\infty_0(\mathcal{M}, S^*\mathcal{M})$.
- Spin scalar product: $\prec \cdot \,|\, \cdot \succ_x : C^\infty_{sc}(\mathcal{M}, S\mathcal{M}) \times C^\infty_{sc}(\mathcal{M}, S\mathcal{M}) \to \mathbb{C}$
$$\prec \psi_m | \varphi_m \succ_x := \psi_m(x)^\dagger \gamma^0 \varphi_m(x).$$
- **Space-time inner product**: $<\cdot|\cdot> \ : \ C^\infty_{sc}(\mathcal{M}, S\mathcal{M}) \times C^\infty_0(\mathcal{M}, S\mathcal{M}) \to \mathbb{C}$
$$<\psi_m | \varphi_c> = \int_{\mathbb{R}^4} \prec \psi_m | \varphi_c \succ_x \, d^4x \, .$$
- **Scalar product**: $(\cdot|\cdot)_m\big|_t : C^\infty_{sc}(\mathcal{M}, S\mathcal{M}) \times C^\infty_{sc}(\mathcal{M}, S\mathcal{M}) \to \mathbb{C}$
$$(\varphi_m | \psi_m)_m\big|_t := 2\pi \int_{\mathbb{R}^3} \prec \psi_m | \gamma^0 \varphi_m \succ|_{(t,\vec{x})} \, d^3x.$$
- Hilbert space $\mathscr{H}_m := \left(C^\infty_{sc}(\mathcal{M}, S\mathcal{M}), (\cdot|\cdot)_m \right)$

An Old Story II: CAR Algebra and Quasi-Free States

- Space of pairs of spinorial test functions: $\mathfrak{D} := C_0^\infty(M, SM) \oplus C_0^\infty(M, S^*M)$.
- Scalar product: $(\cdot\,,\cdot)_{\mathfrak{D}} : \mathfrak{D} \times \mathfrak{D} \to \mathbb{R}^+$ defined as

$$\big(f \oplus g \,\big|\, a \oplus b\big)_{\mathfrak{D}} := <f \,|\, \tilde{\mathcal{K}}_m a> + <\tilde{\mathcal{K}}_m^* b^* \,|\, g^*>$$

- Involution map: $\Gamma : \mathfrak{D} \to \mathfrak{D}$ such that $\Gamma^2 = \mathbb{1}$ and $\big(\Gamma h_1, \Gamma h_2\big)_{\mathfrak{D}} = (h_2, h_1)_{\mathfrak{D}}$.

The **field algebra** $\mathcal{F}$: unital $*$-algebra generated by the abstract elements $B(h)$ with $h \in \mathfrak{D}$ which satisfy:

(i) Linearity: $B(\alpha f \oplus g + m \oplus \beta n) = \alpha B(f \oplus g) + \beta B(m \oplus n)$

(ii) Hermiticity: $B(f \oplus g)^* = B\big(\Gamma(f \oplus g)\big)$

(iii) Dynamics: $B\big((\mathcal{D} - m)f \oplus (\mathcal{D}^* - m)g\big) = 0$ for all $f \oplus g \in \mathfrak{D}$

(iv) CARs: $\big\{B(f \oplus g)^*, B(m \oplus n)\big\} = (f \oplus g \,|\, m \oplus n)_{\mathfrak{D}} \cdot 1_{\mathcal{F}}$.

A **quasi-free state** $\omega : \mathcal{F} \to \mathbb{C}$ if $\omega_{2n+1}(h_1, \dots, h_{2n+1}) = 0$ and

$$\omega_{2n}(h_1, \dots, h_{2n}) = \sum_{\sigma \in S'_{2n}} (-1)^{\mathrm{sign}(\sigma)} \prod_{i=1}^{n} \omega_2\big(h_{\sigma(2i-1)}, h_{\sigma(2i)}\big).$$

Quasi-Free States and Projection Operators

Lemma 3.3 (*H. Araki: On quasifree states of CAR and Bogoliubov automorphisms. (1970/71).*)

Let R be a bounded symmetric operator on $(\mathscr{H}_{\mathfrak{D}}, (.|.)_{\mathfrak{D}})$ with the following properties

(a) $R + \Gamma R \Gamma = \mathbb{1}$,

(b) $0 \leq R = R^* \leq \mathbb{1}$.

Then there exists a unique quasi-free state ω on $\mathcal{F}$ such that

$$\omega\big(B(h)^* B(\tilde{h})\big) = (h \,|\, R\, \tilde{h})_{\mathfrak{D}} \qquad \text{for all } h, \tilde{h} \in \mathscr{H}_{\mathfrak{D}} .$$

- Our motto will be $\Longrightarrow$ "Split the $\mathscr{H}$ilbert space!"

(...but how?)

The Fermionic Projector in a Strip of Space-Time

[*F. Finster and M. Reintjes: A non-perturbative construction of the fermionic projector on globally hyperbolic manifolds I – Space-times of finite lifetime, arXiv:1301.5420 [math-ph], to appear in Adv. Theor. Math. Phys. (2015).*]

- Begin with a simple example, to explain the basic idea:

$$\Omega \subset (-T, T) \times \Sigma \subset \mathcal{M}.$$

- As before the scalar product: $(\cdot|\cdot)_m : C^\infty_{sc}(\Omega, S\,\Omega) \times C^\infty_{sc}(\Omega, S\,\Omega) \to \mathbb{C}$

$$(\varphi_m|\psi_m)_m := 2\pi \int_\Sigma \prec\!\psi_m|\gamma^0\varphi_m\!\succ|_{(t,\vec{x})}\, d\mu_\Sigma.$$

- $\mathscr{H}_m := \big(C^\infty_{sc}(\Omega, S\,\Omega), (\cdot|\cdot)_m\big)$ is an Hilbert space.

- Now the space-time inner product: $<\!\cdot|\cdot\!> \;:\; C^\infty_{sc}(\Omega, S\,\Omega) \times C^\infty_{sc}(\Omega, S\,\Omega) \to \mathbb{C}$

$$<\!\psi_m|\varphi_m\!> = \int_\Omega \prec\!\psi_m|\varphi_m\!\succ_x d\mu_\Omega \qquad \textit{(well defined)}$$

$$|<\!\varphi_m|\psi_m\!>| \leq c\, \|\varphi_m\|_m\, \|\psi_m\|_m \qquad \textit{(bounded)}.$$

- Via **Riesz representation theorem**

$$<\!\varphi_m|\psi_m\!> = (\varphi_m\,|\,\tilde{S}\,\psi_m)_m.$$

- From the spectral theorem, we can construct an orthonormal projector

$$\chi^{\pm}(\tilde{\mathcal{S}}) = \int_{\sigma} \chi^{\pm}(\lambda)\, dE_{\lambda}.$$

- Finally we can obtain a quasi-free state in the all the space-time via:

$$\mathcal{P} := \chi^{+}(\tilde{\mathcal{S}}) \cdot \tilde{\mathcal{K}}.$$

Problems:

- For $T = \infty$ the space-time inner product is not well defined,
- $\mathcal{P}(x, y)$ is in general not Hadamard!
  [*C. Fewster and B. Lang: Pure quasifree states of the Dirac field from the fermionic projector, arXiv:1408.1645 [math-ph].*]

Mass Oscillation Property

[F. Finster and M. Reintjes: A non-perturbative construction of the fermionic projector on globally hyperbolic manifolds II – Space-times of infinite lifetime, arXiv:1312.7209 [math-ph].]

- Families of solutions of families of Dirac equations:

$$\Psi := (\psi_m)_{m \in \mathrm{I}=(m_L,m_R)} \in \mathscr{H}^\infty .$$

- New **scalar product**: $(\cdot \,, \cdot) : \mathscr{H}^\infty \times \mathscr{H}^\infty \to \mathbb{C}$

$$(\Psi|\Phi) = \int_\mathrm{I} (\psi_m|\varphi_m)_m \, dm = 2\pi \int_\mathrm{I} \int_{\mathbb{R}^3} \prec\psi_m|\gamma^0\varphi_m\succ|_{(t,\vec{x})} \, d^3x \, dm.$$

- Integration over mass as operator

$$\mathfrak{p} \, : \, \mathscr{H}^\infty \to C^\infty_{\mathrm{sc}}(\mathcal{M}, S\mathcal{M}) \, , \quad \mathfrak{p}\Psi = \int_I \psi_m \, dm \, .$$

- "$\mathfrak{p}$" **space-time inner product:**

$$<\mathfrak{p} \cdot | \mathfrak{p} \cdot> \, : \, \mathscr{H}^\infty \times \mathscr{H}^\infty \to \mathbb{C}$$

Q: *Is the new space-time inner product bounded?*

- The Dirac operator has the **strong mass oscillation property** in I$= (m_L, m_R)$ if there exists a constant $c > 0$ such that

$$\left| <\mathfrak{p}\Psi \mid \mathfrak{p}\Phi> \right| = \left| \int_{\mathbb{R}^4} \prec \mathfrak{p}\Psi \mid \mathfrak{p}\Phi \succ_x d^4x \right| \leq c \int_I \|\psi_m\|_m \, \|\varphi_m\|_m \, dm$$

  for all families of solutions $\Psi, \Phi \in \mathscr{H}^\infty$.

- Then there exists a family of linear operators $(\tilde{\mathcal{S}}_m)_{m \in \mathrm{I}}$ with $\tilde{\mathcal{S}}_m \in \mathrm{L}(\mathscr{H}_m)$ which are uniformly bounded $\sup_{m \in I} \|\tilde{\mathcal{S}}_m\| < \infty$, such that

$$<\mathfrak{p}\Psi|\mathfrak{p}\Phi> = \int_I (\psi_m \,|\, \tilde{\mathcal{S}}_m \, \varphi_m)_m \, dm \qquad \text{for all } \Psi, \Phi \in \mathscr{H}^\infty .$$

- The operator $\tilde{\mathcal{S}}_m$ is **uniquely determined** for every $\boldsymbol{m} \in I$ by demanding that for all $\Psi, \Phi \in \mathscr{H}^\infty$, the functions $(\psi_m|\tilde{\mathcal{S}}_m\varphi_m)_m$ are continuous in m.

The Fermionic Projector in the Minkowski Vacuum $(\mathcal{B} = 0)$

- Family of solutions of the Dirac equations $\Psi = (\psi_m)_{m\in I} \in \mathscr{H}^\infty$ in momentum space via the Fourier transform of a solution :

$$\psi_m(k) = 2\pi\, \delta(k^2 - m^2)\, \varepsilon(k^0)\, (\not{k} + m)\, \gamma^0 \hat{\psi}^0_m(\vec{k})\,.$$

- After integration over m

$$(\mathfrak{p}\Psi)(k) = 2\pi\, \chi_I(m)\, \frac{1}{2m}\, \varepsilon(k^0)\, (\not{k} + m)\, \gamma^0 \hat{\psi}^0_m(\vec{k})\Big|_{m=\sqrt{k^2}}$$

we compute the "$\mathfrak{p}$" space-time inner product

$$<\mathfrak{p}\psi|\mathfrak{p}\varphi> = \int \frac{d^4k}{4\pi^2}\, \chi_I(m)\, \frac{1}{2m} \prec\gamma^0 \hat{\psi}^0_m(\vec{k})\,|\,(\not{k} + m)\, \gamma^0 \hat{\varphi}^0_m(\vec{k})\succ\Big|_{m=\sqrt{k^2}}\,.$$

- Reparametrizing the k^0-integral as an integral over m, we obtain

$$<\mathfrak{p}\psi|\mathfrak{p}\varphi> = \frac{1}{4\pi^2} \int_I dm \int_{\mathbb{R}^3} \frac{d^3k}{2\,|k^0|} \prec\gamma^0 \hat{\psi}^0_m(\vec{k})\,|\,(\not{k}+m)\, \gamma^0 \hat{\varphi}^0_m(\vec{k})\succ\Big|_{k^0=\pm\sqrt{|\vec{k}|^2+m^2}}.$$

- Using the Schwarz inequality and applying Plancherel's theorem

$$|<\mathfrak{p}\psi|\mathfrak{p}\varphi>| \leq \frac{1}{4\pi^2}\int_I dm \int_{\mathbb{R}^3} \|\hat{\psi}^0_m(\vec{k})\| \, \|\hat{\varphi}^0_m(\vec{k})\| d^3k \leq 2\pi \int_I \|\psi_m\|_m \|\varphi_m\|_m dm.$$

- Then exists an operator $\mathcal{S}_m$ uniquely determined for every $m \in I$:

$$\mathcal{S}_m(\vec{k}) := \sum_{k^0 = \pm\omega(\vec{k})} \frac{\not{k} + m}{2\,\omega(\vec{k})}\,\gamma^0\,.$$

- Applying the spectral theorem:

$$\chi^+(\mathcal{S}_m) = \Theta(k^0).$$

- Then the **fermonic projector** in momentum space is

$$\mathcal{P} = \chi^+(\mathcal{S}_m)\mathcal{K}_m = \underbrace{\Theta(k^0)}_{\text{Select the positive frequencies}} \cdot\; 2\pi\,\delta(k^2 - m^2)\,\varepsilon(k^0)\,(\not{k} + m)\,\gamma^0$$

External Potential in Minkowski space-time

- If the external potential $\mathscr{B}$ satisfies the conditions

$$|\mathscr{B}(t)|_{C^2} \leq \frac{c}{1+|t|^{2+\varepsilon}}$$

then the strong mass oscillation property holds.

- For every $\Psi, \Phi \in \mathscr{H}^\infty$,

$$<\mathfrak{p}\Psi|\mathfrak{p}\Phi> = \int_I (\psi_m \,|\, \tilde{\mathcal{S}}_m \,\varphi_m)_m \,dm\,,$$

where $\tilde{\mathcal{S}}_m : \mathscr{H}_m \to \mathscr{H}_m$ are bounded linear operators which act on the wave functions at time t_0 by

$$\begin{aligned}\tilde{\mathcal{S}}_m = \mathcal{S}_m &+ \frac{i}{2}\int_{-\infty}^{\infty} \varepsilon(t-t_0)\left[\mathcal{S}_m\, U_m^{t_0,t}\,\gamma^0 \mathscr{B}(t)\,\tilde{U}_m^{t,t_0} - \tilde{U}_m^{t_0,t}\,\gamma^0 \mathscr{B}(t)\,\mathcal{S}_m\, U_m^{t,t_0}\right] dt \\ &- \frac{1}{2}\left(\int_{t_0}^{\infty}\int_{t_0}^{\infty} + \int_{-\infty}^{t_0}\int_{-\infty}^{t_0}\right) \tilde{U}_m^{t_0,t}\,\gamma^0 \mathscr{B}(t)\,\mathcal{S}_m\, U_m^{t,t'}\,\gamma^0 \mathscr{B}(t')\,\tilde{U}_m^{t',t_0}\,dt\,dt'\end{aligned}$$

Using PDE methods, we worked hard!

(This is the technical core of the paper)

- We decompose $\tilde{\mathcal{S}}_m$ with respect to the above frequency splitting,

$$\tilde{\mathcal{S}}_m = \tilde{\mathcal{S}}^{\mathrm{D}} + \Delta\tilde{\mathcal{S}}\,, \qquad \text{where} \qquad \tilde{\mathcal{S}}^{\mathrm{D}} := \tilde{\mathcal{S}}^+_+ + \tilde{\mathcal{S}}^-_- \quad \text{and} \quad \Delta\tilde{\mathcal{S}} := \tilde{\mathcal{S}}^+_- + \tilde{\mathcal{S}}^-_+\,.$$

- Under the assumption

$$\int_{-\infty}^{\infty} |\mathscr{B}(t)|_{C^0}\, dt < \sqrt{2} - 1$$

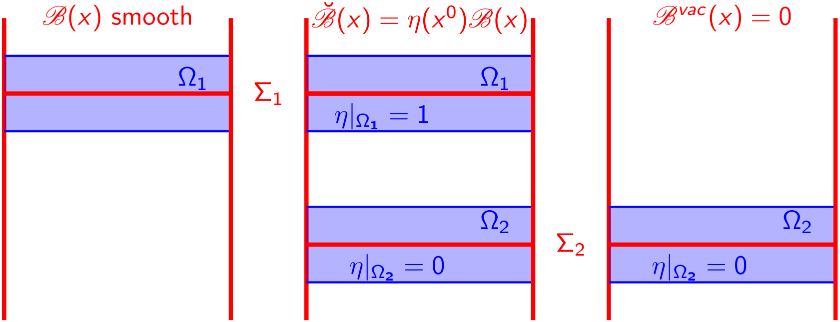
the operators $\chi^\pm(\tilde{\mathcal{S}}_m)$ have the representations

$$\chi^\pm(\tilde{\mathcal{S}}_m) = \chi^\pm(H) + \underbrace{\frac{1}{2\pi i} \oint_{\partial B_{\frac{1}{2}}(\pm 1)} (\tilde{\mathcal{S}}_m - \lambda)^{-1}\, \Delta\tilde{\mathcal{S}}\, (\tilde{\mathcal{S}}^{\mathrm{D}} - \lambda)^{-1}\, d\lambda}_{\text{integral operator with smooth kernel}}\,,$$

- The fermionic projector is given by

$$\mathcal{P} = \chi^+(\tilde{\mathcal{S}}_m)\, \tilde{\mathcal{K}}_m = \chi^+(H)\, \tilde{\mathcal{K}}_m + \textit{ smooth contribution.}$$

Hadamard States: Three Different External Potentials



$\mathcal{P} = \chi^+(H)\,\tilde{\mathcal{K}}_m + \text{(smooth)}$ $\quad\breve{\mathcal{P}} = \chi^+(H)\,\breve{\mathcal{K}}_m + \text{(smooth)}$ $\quad\mathcal{P}^{\text{vac}} = \chi^+(H)\,\mathcal{K}_m$

- $\mathscr{P}^{\text{vac}}(x,y)$ is Hadamard in Ω_0 and then in the whole spacetime.
- $\breve{\mathscr{P}}(x,y) - \mathscr{P}^{\text{vac}}(x,y) \in C^\infty(\mathbb{R}^4)$ for all $x, y \in \Omega_0$, since $\breve{k}_m(x,y) \equiv k_m(x,y)$.
- $\breve{\mathscr{P}}(x,y)$ is Hadamard in Ω_0 and then in the whole spacetime.
- $\mathscr{P}(x,y) - \breve{\mathscr{P}}(x,y) \in C^\infty(\mathbb{R}^4)$ for all $x, y \in \Omega_1$, since $\tilde{k}_m(x,y) \equiv \breve{k}_m(x,y)$.
- $\mathscr{P}(x,y)$ is **Hadamard** in Ω_1 and then in the whole spacetime.

Conclusions

What we know:

- To every projector it is possible to associate an algebraic state.
- The construction holds also in the presence of an external potential.
- The state is Hadamard and when $\mathscr{B} = 0$ is the Poincaré vacuum.

Benefit:

- This technique works without symmetries.
- Allows to construct states for massive particles.

Future investigations:

- in which space-time the Dirac operator satisfies the mass oscillation property,
- if the states constructed turn out to be Hadamard.