A taste of microlocal analysis on supermanifold

Simone Murro

Department of Mathematics
University of Regensburg

Microlocal analysis: a tool to explore a quantum world

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Motivations and goals of my talk

- Supersymmetric QFTs are interesting because of their unexpected renormalization properties

  $\rightsquigarrow$ Non-renormalization theorems [Grisaru, Rocek, Siegel; Seiberg; ...]

- Usual framework is very restrictive: super-QFTs on super-Minkowski space

Q: Do the non-renormalization theorems survive on curved supermanifolds?

A: We don't know yet! Analyzing this question requires heavy machinery such as perturbative locally covariant QFT [Brunetti, Fredenhagen, ...]

- Up to now, the structure of supersymmetry transformations in locally covariant QFT is under controll [Hack, Hanisch, Schenkel]

- **The goal of my talk:**

  1. Introduce a supergeometric generalization of the wavefront set
  2. Generalize to supermanifolds the ordinary pullback theorem for distributions

Outline of the Talk

- Basic aspects of supermanifolds
- The polarization bundle
- Super pseudodifferential operators
- The super wavefront set
- The pullback theorem for superdistributions

Based on:

- C. Dappiaggi, H. Gimperlein, S. M., A. Schenkel, (arXiv:1512.07823)

Basic aspects of supermanifolds

Equivalent definition of smooth manifolds:

1 A topological space M which locally looks like $\mathbb{R}^m$ with smooth transition maps

2 A sheaf $\mathcal{M} = (M, \mathcal{O}_M)$ of commutative algebras which locally looks like $(\mathbb{R}^m, C^\infty_{\mathbb{R}^m})$

Def: A supermanifold is a sheaf $\mathcal{M} = (M, \mathcal{O}_M)$ of supercommutative superalgebras which locally looks like $\mathbb{R}^{m|n} := (\mathbb{R}^m, C^\infty_{\mathbb{R}^m} \otimes \bigwedge^\bullet \mathbb{R}^n)$

- A morphism $\Phi : \mathcal{M} \to \mathcal{N}$ is now a pair $\left(\phi : M \to N, \Phi^* : \mathcal{O}_N(V) \to \mathcal{O}_M(\phi^{-1}(V))\right)$

Def: We call superdomain $U^{m|n} := (U \subset \mathbb{R}^m, C^\infty_U \otimes \wedge^\bullet \mathbb{R}^n) \subseteq \mathbb{R}^{m|n}$

- The section over every $U \subseteq \mathbb{R}^m$ are given by $C^\infty_U \otimes \wedge^\bullet \mathbb{R}^n =: \mathcal{O}_{\mathbb{R}^{m|n}}(U)$

$$f = \sum_{I \in \mathbb{Z}_2^n} f_I\, \theta^I := \sum_{(i_1,\dots,i_n)\in\mathbb{Z}_2^n} f_{(i_1,\dots,i_n)}\, \theta^{1\,i_1} \cdots \theta^{n\,i_n} ,$$

Ex. Let us consider $\mathbb{R}^{1|1}$ and the superdistribution $u = u_1 + \delta\,\theta$ with $u_1 \in C^\infty(\mathbb{R})$:

u^2 is well defined!

The polarization bundle

Def: The polarization bundle is defined as $\mathcal{P}^* U^{m|n} := T^*U \times \wedge^\bullet \mathbb{C}^n \longrightarrow T^*U$

$$\begin{array}{ccc} \mathcal{P}^* V^{m'|n'}\big|_{T^*_{\phi(x)}V} & \xrightarrow{\quad \mathcal{P}^*\Phi \quad} & \mathcal{P}^* U^{m|n}\big|_{T^*_x U} \\ \Big\downarrow \pi & & \Big\downarrow \pi \\ T^*_{\phi(x)}V & \xrightarrow[\quad T^*\phi \quad]{} & T^*_x U \end{array}$$

- Any superalgebra morphism $\Phi^* : \mathcal{O}_{V^{m'|n'}} \to \mathcal{O}_{U^{m|n}}$ can be factorized uniquely

$$(\Phi^*_V)_j{}^i = \begin{cases} \phi^* \circ (D^\Phi)_j{}^i & \text{if } j-i \geq 0 \text{ even} \\ 0 & \text{else} \end{cases}$$

- We define the mapping $\mathcal{P}^*\Phi$ component-wise by

$$\big(\phi(x), k', \lambda'\big) \longmapsto \begin{cases} \Big(x, T^*\phi(k'), \sigma_{\frac{j-i}{2}}(D^\Phi)_j{}^i(\phi(x), k') \cdot \lambda'\Big) & \text{if } j-i \geq 0 \text{ even} \\ \big(x, T^*\phi(k'), 0\big) & \text{else} \end{cases}$$

- The polarization mapping is compatible with compositions

$$\mathcal{P}^*(\Phi' \circ \Phi) = (\mathcal{P}^*\Phi) \circ (\mathcal{P}^*\Phi')$$

Super pseudodifferential operators

Def: A super pseudodifferential operator is linear map $A : \mathcal{O}_c(U) \to \mathcal{O}(U)$

$$A_j{}^i : C_c^\infty(U) \otimes \wedge^i \mathbb{R}^n \to C^\infty(U) \otimes \wedge^j \mathbb{R}^n$$

are (matrices of) pseudodifferential operators

Def: A super pseudodifferential operator A on $U^{m|n}$ is of order l if

$$s\Psi\mathrm{DO}^l(U^{m|n}) := \left\{ A : C_c^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n \to C^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n : A_j{}^i \in \Psi\mathrm{DO}^{\frac{j-i}{2}+l}(U) \right\} .$$

Prop: Let $A \in s\Psi\mathrm{DO}^l(U^{m|n})$ and $B \in s\Psi\mathrm{DO}^{l'}(U^{m|n})$. Then:

a) $B \circ A \in s\Psi\mathrm{DO}^{l+l'}(U^{m|n})$ and $\sigma_{l+l'}(B \circ A) = \sigma_{l'}(B) \circ \sigma_l(A)$

b) If $\Phi : U^{m|n} \to V^{m|n}$ is a supermanifold isomorphism, then

$$\Phi_V^{*\,-1} \circ A \circ \Phi_V^* \in s\Psi\mathrm{DO}^l(V^{m|n})$$
$$\sigma_l(\Phi_V^{*\,-1} \circ A \circ \Phi_V^*) = (\mathcal{P}^*\Phi^{-1}) \circ \sigma_l(A) \circ (\mathcal{P}^*\Phi) .$$

Example: Wess-Zumino model

- $\mathcal{M} = (M, C^\infty_M \otimes \wedge^\bullet \mathbb{R}^2)$, where M is a smooth 3-dimensional Lorentzian manifold.
- The equation of motion $P : \mathcal{O}_M \to \mathcal{O}_M$ of the 3|2-dimensional Wess-Zumino model

$$P = \begin{pmatrix} m & 0 & -1 \\ 0 & \mathrm{i}\,\nabla\!\!\!/ + m & 0 \\ \Box & 0 & m \end{pmatrix} ,$$

- The operator $P \in s\Psi\mathrm{DO}^1(X)$ is of order 1, and in local coordinates

$$\sigma_1(P)(x,k) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -\gamma^\mu(x)\, k_\mu & 0 \\ -k_\mu k_\nu\, g^{\mu\nu}(x) & 0 & 0 \end{pmatrix}$$

- $\sigma_1(P)(x,k)$ is invertible for all $(x,k) \in T^*M \setminus \mathbf{0}$ which are not null covector

$$\sigma_1(P)(x,k)^{-1} = \begin{pmatrix} 0 & 0 & -\frac{1}{k_\mu k_\nu\, g^{\mu\nu}(x)} \\ 0 & -\frac{\gamma^\mu(x)\, k_\mu}{k_\mu k_\nu\, g^{\mu\nu}(x)} & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

- Because $\sigma_1(P)(x,k)$ is invertible for non-null cotangent vector we call P hyperbolic

The super wavefront set

- The super wavefront set (of order I) of a superdistribution $u \in \mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$

$$\mathrm{sWF}^I(u) := \bigcap_{\substack{A \in s\Psi\mathrm{DO}^I(U^{m|n}) \\ \text{s.t. } Au \text{ smooth}}} \Big\{(x,k,\lambda) \in \widehat{\mathcal{P}}^* U^{m|n} \,:\, \sigma_I(A)(x,k)\big(\lambda\big) = 0\Big\} \subseteq \widehat{\mathcal{P}}^* U^{m|n} \,.$$

where $\pi : \widehat{\mathcal{P}}^* U^{m|n} := \pi^{-1}\big(T^*U \setminus \mathbf{0}\big) \longrightarrow T^*U \setminus \mathbf{0}$

Prop: Let $u = \sum_{I \in \mathbb{Z}_2^n} u_I \, \theta^I \in \mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$ and $A \in s\Psi\mathrm{DO}^I(U^{m|n})$:

a) $\mathrm{sWF}^I(u) = \mathrm{sWF}^{I'}(u)$ for all I, I'

b) $\pi\left(\mathrm{sWF}^I(u) \setminus \big((T^*U \setminus \mathbf{0}) \times \{0\}\big)\right) = \bigcup_{I \in \mathbb{Z}_2^n} \mathrm{WF}(u_I)$

c) $u \in \mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$ is smooth if and only if $\mathrm{sWF}(u) = (T^*U \setminus \mathbf{0}) \times \{0\}$

d) $\mathrm{sWF}(Au) \supseteq \sigma_I(A)\big(\mathrm{sWF}(u)\big) := \big\{\big(x,k,\sigma_I(A)(x,k)\big(\lambda\big)\big) : (x,k,\lambda) \in \mathrm{sWF}(u)\big\}$

e) If $\Phi : U^{m|n} \to V^{m|n}$ is a supermanifold isomorphism

$$\mathrm{sWF}(\Phi_V^*(u)) = \mathcal{P}^*\Phi\big(\mathrm{sWF}(u)\big)$$

Example II: the polarization

- $U^{m|2}$ and the superdistribution $u = v + v\,\theta^1\theta^2$
- The super pseudodifferential operator of order 0

$$Au = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ 0 \\ v \end{pmatrix} = 0$$

- The super principal symbol of order 0 of A reads as

$$\sigma_0(A)(x,k) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,,$$

- The super wavefront set $s\mathrm{WF}(u) \subseteq \left(T^*U \setminus \mathbf{0}\right) \times \left\{\lambda \in \wedge^\bullet \mathbb{C}^2 \,:\, \lambda_{(1,1)} = 0\right\}$
- Our super wavefront sets picks out the leading singularities and assigns higher weight to the components with a lower number of θ-powers.
- This feature generalizes to superdomains in higher odd-dimensions $U^{m|n}$.

The pullback theorem for superdistributions

- The normal set of a smooth map $\phi : U \subseteq \mathbb{R}^m \to V \subseteq \mathbb{R}^{m'}$ is

$$N_\phi := \Big\{ \big(\phi(x), k'\big) \in T^*V \;:\; x \in U\,,\;\; T^*\phi(k') = 0 \Big\} \subset T^*V$$

- Hormander proved that the pullback map $\phi^* : C^\infty(V) \to C^\infty(U)$ admits a *unique continuous extension* to those distributions $u \in \mathcal{D}'(V)$ which satisfy

$$\mathrm{WF}(u) \cap N_\phi = \emptyset.$$

- Let us now consider a supermanifold morphism $\Phi : U^{m|n} \to V^{m'|n'}$
  1. the superalgebra morphism factorized as $\Phi_V^* = \phi^* \circ D^\phi$
  2. $D^\Phi u \in \mathcal{D}'(V) \otimes \wedge^\bullet \mathbb{R}^n$ is *always well-defined*
  3. $\phi^* D^\Phi u \in \mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$ exists if we assume

$$\pi\Big(s\mathrm{WF}(D^\Phi u) \setminus \big((T^*V \setminus \mathbf{0}) \times \{0\} \big) \Big) \cap N_\phi = \emptyset \;,$$

Thm: The pullback map $\Phi_V^* : \mathcal{O}_V \to \mathcal{O}_U$ admits a unique continuous extension to those superdistributions $u \in \mathcal{D}'(V) \otimes \wedge^\bullet \mathbb{R}^{n'}$ which satisfy

$$\Big(\bigcup_{I \in \mathbb{Z}_2^n} \mathrm{WF}\big((D^\Phi u)_I\big) \Big) \cap N_\phi$$

- Another condition which would guarantee the existence of $\Phi_V^* u$ is

$$\pi\Big(s\mathrm{WF}(u) \setminus \big((T^*V \setminus \mathbf{0}) \times \{0\}\big)\Big) \cap N_\phi = \bigcup_{J \in \mathbb{Z}_2^{n'}} \mathrm{WF}(u_J) \cap N_\phi = \emptyset$$

- ... but this is a strong requirement

$$\mathrm{WF}\big((D^\Phi u)_I\big) \cap N_\phi = \mathrm{WF}\Big(\sum_{J \in \mathbb{Z}_2^{n'}} (D^\Phi)_I{}^J u_J\Big) \cap N_\phi$$

$$\subseteq \bigcup_{J \in \mathbb{Z}_2^{n'}} \mathrm{WF}\big((D^\Phi)_I{}^J u_J\big) \cap N_\phi \subseteq \bigcup_{J \in \mathbb{Z}_2^{n'}} \mathrm{WF}(u_J) \cap N_\phi = \emptyset \quad (1)$$

- Consider the supermanifold morphism $\Phi : \{*\} \to U^{m|n}$

$$\Phi_U^* : C^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n \longrightarrow \mathbb{R} \,, \quad f = \sum_{I \in \mathbb{Z}_2^n} f_I \, \theta^I \longmapsto f_{(0,\ldots,0)}(\phi(*))$$

- We can extend Φ_U^* to *all* superdistributions with smooth lowest component $u_{(0,\ldots,0)}$
- Because $N_\phi = T^*_{\phi(*)} U$, *the condition* (1) *is violated* if any u_I is singular at this point
- In contrast, our condition is verified because just involves the lowest component

$$D^\Phi = (1 \ 0 \ \ldots \ 0) \text{ and hence } D^\Phi u = u_{(0,\ldots,0)}$$

The dessert: product of superdistributions

- The super diagonal mapping $\Delta : U^{m|n} \longrightarrow U^{m|n} \times U^{m|n} \simeq (U \times U)^{2m|2n}$
  1. $\widetilde{\Delta} : U \to U \times U$ defined as $x \mapsto (x, x)$
  2. $\Delta^*_{U\times U} : C^\infty(U \times U) \otimes \wedge^\bullet\mathbb{R}^n \otimes \wedge^\bullet\mathbb{R}^n \to C^\infty(U) \otimes \wedge^\bullet\mathbb{R}^n$

$$\Delta^*_{U\times U} = \widetilde{\Delta}^* \circ D^\Delta = (\widetilde{\Delta}^* \otimes \mathrm{id}_{\wedge^\bullet\mathbb{R}^n}) \circ (\mathrm{id}_{C^\infty(U\times U)} \otimes \mu)$$

  3. $N_{\widetilde{\Delta}} = \Big\{ \big((x,x),(k,-k)\big) \in T^*(U \times U) \, : \, (x,k) \in T^*U \Big\}$
- Given $u, v \in \mathcal{D}'(U) \otimes \wedge^\bullet\mathbb{R}^n$, their product (if it exists) is given by

$$u \cdot v := \Delta^*_{U\times U}(u \otimes v) = (\widetilde{\Delta}^* \circ D^\Delta)\Big(\sum_{I,J \in \mathbb{Z}_2^n} u_I \otimes v_J \, (\theta^I \otimes \theta^J) \Big) =$$

$$= \widetilde{\Delta}^* \Big(\sum_{I,J \in \mathbb{Z}_2^n} u_I \otimes v_J \, (\theta^I \, \theta^J) \Big) = \sum_{I,J \in \mathbb{Z}_2^n} \widetilde{\Delta}^*(u_I \otimes v_J) \, (\theta^I \, \theta^J)$$

Cor: The product $u\,v \in \mathcal{D}'(U) \otimes \wedge^\bullet\mathbb{R}^n$ exists whenever

$$\pi\Big(s\mathrm{WF}\big(D^\Delta(u \otimes v)\big) \setminus \big((T^*(U \times U) \setminus \mathbf{0}) \times \{0\}\big) \Big) \cap N_{\widetilde{\Delta}} = \emptyset$$