Linearized gravity and Hadamard states

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General Framework

Is there a quantum theory of General Relativity?

1) String theory, loop quantum gravity.
2) Algebraic approach to quantum gravity.
   [*Brunetti, Fewster, Fredenhagen, Giesel, Majid, Rejzner, Tiemann, . . .*]

Algebraic approach ⟶ **rigorous approach** to QFT

[*Araki, Bär, Brunetti, Buchholz, Dappiaggi, Dimock, Finster, Fredenhagen, Gérard, Ginoux, Haag, Kay, Moretti, Pinamonti, Strohmaier, Verch, Wald, . . .*]

Why is the construction of a Hadamard state important?

1) Evaluation of the *influence of gravitational field* on physical observables.
2) Understand in the algebraic approach *structural problems* of quantum gravity.
   [*Brunetti, Fredenhagen, Rejzner: Quantum gravity from the point of view of locally covariant quantum field theory*]

Outline

- **On the algebraic approach to quantum field theory**
- **Linearized gravity on globally hyperbolic spacetimes**
- **Hadamard states for linearized gravity on asymptotically flat spacetimes**

Based on:

- M. Benini, C. Dappiaggi and S.M. - J. Math. Phys. **55** 082301 (2014)

PART I:

On the algebraic approach to quantum field theory

AQFT I - Scalar field

[C. Bär, N. Ginoux, F. Pfäffle: *Wave Equations on Lorentzian Manifolds and Quantization* - European Mathematical Society (2007)]

Goal: Outline AQFT via a good example!

- $\mathcal{M} \simeq \mathbb{R} \times \Sigma$ is 4-dim is a **globally hyperbolic spacetime**

$$ds^2 = \beta^2 dt^2 - h_t; \quad \beta \in C^\infty(M; \mathbb{R}^+) \text{ and } h_t \in \text{Riem}(\Sigma); \forall t \in \mathbb{R}$$

- $\varphi : M \to \mathbb{R}$ is a **conformally real scalar field**

$$P\varphi = \left(-\Box + \frac{1}{6}R\right)\varphi = 0$$

- $P : C^\infty(M) \to C^\infty(M)$ is **normally hyperbolic** then $\exists\, G^\pm : C^\infty_c(\mathcal{M}) \to C^\infty_{sc}(\mathcal{M})$

$$\text{(i) } P\,G^\pm f = f \qquad \text{(ii) } G^\pm\,P f = f \qquad \text{(iii) } \mathrm{supp}(G^\pm f) \subset J^\pm(\mathrm{supp}(f))$$

- All **dynamical configurations** of a real scalar field are

$$\mathcal{S}ol(\mathcal{M}) = \{\varphi \in C^\infty_{sc}(\mathcal{M}) \,|\, \exists f \in C^\infty_c(M) \text{ and } \varphi = Gf\}$$

AQFT II - Classical Observables

[C. Dappiaggi, G. Nosari, N. Pinamonti: *The Casimir effect from the point of view of algebraic quantum field theory* - Math. Phys. Anal. Geom. **19**, 12 (2016)]

A *classical observable* is an assignment of a real number to each dynamical configuration

$$\forall\varphi \in C^\infty(\mathcal{M}) \text{ there exists } \alpha \in C^\infty_c(\mathcal{M}) \mapsto O_\alpha(\varphi) := \int_M d\mu_g \varphi(x)\alpha(x)$$

- **Space of classical observables**

$$\mathcal{E}(\mathcal{M}) := \left\{ O_{[\alpha]} : \mathcal{S}ol(\mathcal{M}) \to \mathbb{R} \,\middle|\, \forall\varphi\, \exists [\alpha] \in \frac{C^\infty_c(\mathcal{M})}{P[C^\infty_c(\mathcal{M})]} \mapsto O_{[\alpha]}(\varphi) := \int_M d\mu_g \varphi(x)\alpha(x) \right\}$$

We have identified classical observables as the vector space $\mathcal{E}(\mathcal{M}) \simeq \frac{C^\infty_c(\mathcal{M})}{P[C^\infty_c(\mathcal{M})]}$

Why do we believe it is the right choice? Paradigm is:

- $\mathcal{E}(\mathcal{M})$ is **separating**: $\forall\ \varphi, \varphi' \in \mathcal{S}ol(\mathcal{M})$, $\exists\, [\alpha] \in \mathcal{E}(\mathcal{M})$ s. t. $O_{[\alpha]}(\varphi) \neq O_{[\alpha]}(\varphi')$
- $\mathcal{E}(\mathcal{M})$ is **optimal**: $\forall\ [\alpha], [\alpha'] \in \mathcal{E}(\mathcal{M})$, $\exists\, \varphi \in \mathcal{S}ol(\mathcal{M})$ s. t. $O_{[\alpha]}(\varphi) \neq O_{[\alpha']}(\varphi)$

AQFT III - Algebra of Observables

[J. Dimock: *Algebras of Local Observables on a Manifold* - Commun. Math. Phys. **77**, 219 (1980)]

Gaol: From $\mathcal{E}(\mathcal{M};\mathbb{C}) = \mathcal{E}(\mathcal{M}) \otimes \mathbb{C}$ build the "algebra of fields"

(1) Construct the unital Borchers-Uhlmann $*$-algebra:

$$\mathscr{A} = \bigoplus_{n=0}^{\infty} \mathcal{E}(\mathcal{M};\mathbb{C})^{\otimes n}$$

where $\mathcal{E}(\mathcal{M};\mathbb{C})^0 = \mathbb{C}$ and the $*$-operation is complex conjugation

(2) Construct the ideal $\mathscr{I}(\mathcal{M}) \subset \mathscr{A}(\mathcal{M})$ generated by elements of the form

$$[\alpha] \otimes [\alpha'] - [\alpha'] \otimes [\alpha] - \imath\, G([\alpha],[\alpha'])\mathbb{1}\,,$$

where $\mathbb{1}$ is the unit in $\mathscr{A}(\mathcal{M})$ and

$$G([\alpha],[\alpha']) \doteq (\alpha \mid G\alpha') = \int_M d\mu_g \alpha(x) G(\alpha')(x)$$

(3) Define the **Algebra of Fields** $\mathscr{F}(\mathcal{M}) \doteq \frac{\mathscr{A}(\mathcal{M})}{\mathscr{I}(\mathcal{M})}$

AQFT IV - Hadamard States

[M. J. Radzikowski: *Micro-local approach to the Hadamard condition in QFT on curved space-time* - Commun. Math. Phys. **179**, 529 (1996)]

- An algebraic **state** $\omega : \mathscr{F}(\mathcal{M}) \to \mathbb{C}$ is a linear functional such that:

$$\omega(\mathbb{1}) = 1 \quad \text{(normalized)} \qquad \omega(a^* a) \geq 0 \text{ (positive)}$$

- Notice that choosing a state $\omega : \mathscr{F}(\mathcal{M}) \to \mathbb{C}$ is equivalent to assigning

$$\omega_n(\alpha_1, \dots, \alpha_n) \qquad \forall n \in \mathbb{N} \text{ and } \forall \alpha_i \in C_c^\infty(\mathcal{M})$$

Which criteria to choose a *physical* state on a curved spacetime?

(1) **Quasifree** ($\omega_{2n+1} \equiv 0$ and ω_{2n} is determined by ω_2),

(2) ***Hadamard:*** $WF(\omega_2) = \left\{(x, y, \xi_x, \xi_y) \in T^*M^{\otimes 2} \setminus 0 \,\middle|\, (x, \xi_x) \sim (y, -\xi_y),\ \xi_x \triangleright 0\right\}$

- same ultraviolet behaviour as the vacuum state,
- quantum fluctuations of all observables are finite,
- covariant construction of Wick polynomials to deal with interactions.

PART II:

Linearized gravity on globally hyperbolic spacetimes

Linearized Einstein equations I

- On a globally hyperbolic spacetime $(\mathcal{M}, g)$ such that $Ric(g) = 0$,

$$(\mathscr{L}h)_{\mu\nu} = -\frac{1}{2}(\Box h_{\mu\nu} - g_{\mu\nu}\Box h^{\alpha}{}_{\alpha}) + R^{\alpha}{}_{\mu\nu}{}^{\beta} h_{\beta\alpha} + \nabla_{(\mu}\nabla^{\alpha} h_{\nu)\alpha} - \frac{1}{2}\nabla_{\mu}\nabla_{\nu} h^{\alpha}{}_{\alpha} - \frac{1}{2} g_{\mu\nu}\nabla^{\alpha}\nabla^{\beta} h_{\alpha\beta} = 0\,. \tag{1}$$

- This system exhibits a **gauge symmetry**:

$$h \sim h' \quad \Longleftrightarrow \quad \exists \chi \in \Gamma(T^*\mathcal{M}) \text{ such that } h' = h + \nabla_s \chi$$

where $(\nabla_s \chi)_{\mu\nu} = \frac{1}{2}(\nabla_\mu \chi_\nu + \nabla_\nu \chi_\mu)$

- **Gauge equivalence class of solutions** $\mathcal{S}ol(\mathcal{M})/\mathcal{G}$ where

$$\mathcal{S}ol(\mathcal{M}) \doteq \{h \in \Gamma_{sc}(\otimes_s^2 T^*\mathcal{M}) \,|\, (\mathscr{L}h)_{\mu\nu} = 0\}.$$

$$\mathcal{G}(\mathcal{M}) = \{\mathcal{L}_\xi g \in \Gamma_{sc}(\otimes_s^2 T^*\mathcal{M}) \,|\, \xi \in \Gamma_{sc}(T^*\mathcal{M})\}$$

Linearized Einstein equations II - de Donder gauge

Lemma

For every $[h] \in \mathcal{S}ol(\mathcal{M})/\mathcal{G}$, there exists a representative h such that

$$\begin{cases} \mathrm{P}h = (\Box - 2\mathrm{Riem})\, \mathrm{I}\, h = 0 \\ \mathrm{div}\, \mathrm{I}\, h = 0 \end{cases}$$

where $(\mathrm{div}\,(h))_\mu = \nabla^\nu h_{\mu\nu}$ and $(\mathrm{I}\, h)_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h^a_a$

Notice that:

- $\tilde{P} = \Box - 2\mathrm{Riem}$ is normally hyperbolic $\Rightarrow$ causal propagator: $\tilde{P} \circ G_{\tilde{P}} = G_{\tilde{P}} \circ \tilde{P} = 0$
- the trace reversal I does not spoil hyperbolicity, let $G_P = G_{\tilde{P}} \circ I$

$$G_P \circ P = P \circ G_P = 0$$

- Since $\mathrm{div}\, \mathrm{I}\, G^{\pm}_P = G^{\pm}_{\Box} \mathrm{div}$, the fixing is not complete:

$$\forall\, [h] \text{ there exist } h', h \text{ such that } h' - h = \nabla_s \chi \text{ where } \Box\chi = 0$$

Linearized Einstein equations III - de Donder gauge

Theorem

There exists a 1:1 correspondence $\frac{\mathcal{S}ol(\mathcal{M})}{\mathcal{G}} \longleftrightarrow \frac{\mathrm{Ker}_c(\mathrm{div})}{\mathrm{Im}_c(\mathscr{L})}$

$$\mathrm{Ker}_c(\mathrm{div}) \doteq \{\varepsilon \in \Gamma_c(\otimes_s^2 T^*\mathcal{M}) \,|\, \mathrm{div}(\varepsilon) = 0\}$$

$$\mathrm{Im}_c(\mathscr{L}) \doteq \{\varepsilon \in \Gamma_c(\otimes_s^2 T^*\mathcal{M}) \,|\, \varepsilon = \mathscr{L}\gamma \text{ with } \gamma \in \Gamma_c(\otimes_s^2 T^*\mathcal{M})\}$$

- The isomorphism is realized by the map

$$\frac{\mathrm{Ker}_c(\mathrm{div})}{\mathrm{Im}_c(\mathscr{L})} \ni [\varepsilon] \mapsto [G_P(\varepsilon)] \in \frac{\mathcal{S}ol(\mathcal{M})}{\mathcal{G}}$$

Important: There is a topological obstruction in implementing the TT gauge: whenever the Cauchy surface is compact!

[C. J. Fewster, D. S. Hunt : *Quantization of linearized gravity in cosmological vacuum spacetimes* - Rev. Math. Phys. **25**, 1330003 (2013)]

Classical observables and the algebra of fields

- Mimicking the case of the scalar field, the **classical observables** are

$$\mathcal{E}(\mathcal{M}) = \frac{\mathcal{E}^{inv}}{\mathrm{Im}_c(\mathscr{L})}\,, \quad \text{with } \mathcal{E}^{inv} \doteq \{\varepsilon \in \Gamma_c(\otimes_s^2 T^*\mathcal{M}) \,|\, \mathrm{div}(\varepsilon) = 0\}$$

- The *Borchers-Uhlmann algebra* $\mathcal{A}(\mathcal{M})$ is defined as follow:

$$\mathcal{A}(\mathcal{M}) = \bigoplus_{n=0}^{\infty} \mathcal{E}(\mathcal{M};\mathbb{C})^{\otimes n}$$

- Take a quotient of $\mathcal{A}(\mathcal{M})$ by the ideal $\mathcal{I}$ generated by

$$[\varepsilon]\otimes[\varepsilon'] - [\varepsilon']\otimes[\varepsilon] - \imath G([\varepsilon],[\varepsilon'])\mathbb{1}\,, \qquad G([\varepsilon],[\varepsilon']) = \int_{\mathcal{M}} d\mu_g \alpha(x) G(\alpha')(x)$$

- The resulting **algebra of fields**: $\mathcal{F}(\mathcal{M}) \doteq \mathcal{A}(\mathcal{M})/\mathcal{I}$.

PART III:

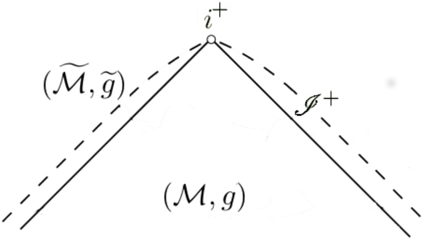
Hadamard states for linearized gravity on asymptotically flat spacetimes

Algebraic holography

[Dappiaggi, Pinamonti, Moretti: *Rigorous steps towards holography in asymptotically flat spacetimes* - Rev. Math. Phys. **18**, 349 (2006)]

1) Encode the information of a QFT defined on the manifold into a counterpart living on the boundary.

2) Asymptotically flat spacetimes:



(i) $(\mathcal{M}, g) \hookrightarrow (\widetilde{\mathcal{M}}, \widetilde{g} = \Omega^2 g) \mapsto (\widetilde{\mathcal{M}}, \omega\widetilde{g})$

(ii) Ω has smooth extension on $\widetilde{M}$ and $\Omega|_{\mathscr{I}^+ \cup \imath^+} \equiv 0$, $d\Omega|_{\mathscr{I}^+} \neq 0$ and $d\Omega|_{\imath^+} = 0$

(iii) $\mathscr{I}^+$ is lightlike 3D submanifold of $\widetilde{\mathcal{M}}$

(iv) $n^\mu = \tilde{\nabla}^\mu \Omega$ vector field tangent to $\mathscr{I}^+$

(v) $(\mathscr{I}^+, q, n) \leftrightsquigarrow$ *universal structure*

(vi) ***the BMS group*** $\varphi : \mathscr{I}^+ \to \mathscr{I}^+$

$$\mathscr{I}^+ \mapsto \mathscr{I}^+, \quad q \mapsto \omega^2 q, \quad n \mapsto \omega^{-1} n$$

The algebra of fields on $\mathscr{I}^+$

- Inspired by Ashtekar & Magnon (1982) define

$$\mathcal{E}(\mathscr{I}^+) = \{\lambda \in \Gamma(\otimes_s^2 T^*\mathscr{I}^+) \,|\, \lambda_{\mu\nu} n^\mu = 0,\ \lambda_{\mu\nu} q^{\mu\nu} = 0,\\ (\lambda,\lambda) < \infty,\ (\partial_\mu\lambda, \partial_\mu\lambda) < \infty\}$$

where $n^\mu = \nabla^\mu \Omega$, $q^{\mu\nu}$ satisfies $q^{\mu\nu} q_{\mu\alpha} q_{\nu\beta} = q_{\alpha\beta}$ and

$$(\lambda,\lambda) = \int_{\mathscr{I}^+} d\mu_{\mathscr{I}^+} \lambda_{\mu\nu} \lambda_{\alpha\beta} q^{\mu\nu} q^{\alpha\beta}$$

- The **algebra of fields** $\mathcal{F}(\mathscr{I}^+)$ is defined as follow:

$$\mathcal{F}(\mathscr{I}^+) \doteq \frac{\bigoplus_{n=0}^{\infty} \mathcal{E}(\mathcal{M};\mathbb{C})^{\otimes n}}{\mathcal{I}(\mathscr{I}^+)}$$

where $\mathcal{I}$ is generated by $\lambda \otimes \lambda' - \lambda' \otimes \lambda - \imath \sigma_{\mathscr{I}^+}(\lambda,\lambda')\mathbb{1}$

$$\sigma_{\mathscr{I}^+}(\gamma_1,\gamma_2) = \int_{\mathscr{I}^+} \Big((\gamma_1)^{\mu\nu} \partial_n (\gamma_2)_{\mu\nu} - (\gamma_2)^{\mu\nu} \partial_n (\gamma_1)_{\mu\nu} \Big)\ d\ell \wedge dS^2(\vartheta,\varphi)$$

Bulk to Boundary correspondence I

Goal: Project an element of $\mathcal{E}(\mathcal{M})$ to $\mathcal{E}(\mathscr{I}^+)$

- *First difficulty*: $(\mathcal{M}, g) \mapsto (\tilde{M}, \Omega^2 g)$ transform $\mathscr{L}$ into an operator with terms proportional to Ω^{-n} ... divergences on $\mathscr{I}^+$ since $\Omega \equiv 0$
- *Solution?* If dim $\mathcal{M} > 4$ the TT-gauge saves the day, while dim $\mathcal{M} = 4$ you need **Geroch-Xanthopoulos gauge**

Big Problem: Topological obstruction to implementing the G-X gauge

Theorem

- Let $h' = h + \nabla_s \chi$, $\chi \in \Gamma(T^* M)$. Then $\tau_{\mu\nu} = \Omega h'_{\mu\nu}$ is in the GX-gauge iff

$$\nabla^\mu \nabla_{[\mu} \chi_{\nu]} = -v_\nu(h) \qquad v_\nu(h) = \nabla^\mu h_{\mu\nu} - \nabla_\nu h \;\; \text{(co-exact)}$$

- Let $h = G_P(\varepsilon)$. Then $v_\nu(h)$ is co-exact iff $\mathrm{Tr}(\varepsilon) = g^{\mu\nu}\varepsilon_{\mu\nu}$ is co-exact.

Bulk to Boundary correspondence II

Definition

We say that $[\varepsilon] \in \mathcal{E}(\mathcal{M})$ is a **radiative observable** if there exists a representative $\varepsilon \in [\varepsilon]$ whose trace is co-exact. The collection of all these observables is $\mathcal{E}^{rad}(\mathcal{M}) \subseteq \mathcal{E}(\mathcal{M})$

Big Question: Can $\mathcal{E}^{rad}(\mathcal{M}) = \mathcal{E}(\mathcal{M})$?

Proposition

$\mathcal{E}^{rad}(\mathcal{M}) = \mathcal{E}(\mathcal{M})$ on Minkowski spacetime but $\mathcal{E}^{rad}(\mathcal{M}) \subset \mathcal{E}(\mathcal{M})$ on any asymptotically flat, globally hyperbolic spacetime whose Cauchy surface has a $\mathbb{S}^1$ factor.

- The map $\Gamma : \mathcal{E}^{rad}(\mathcal{M}) \to \mathcal{E}(\mathscr{I}^+)$ defined by

$$G([\varepsilon],[\varepsilon']) = \sigma_{\mathscr{I}}(\Gamma[\varepsilon],\Gamma[\varepsilon'])$$

induces an injective $*$-**homomorphism** $\iota : \mathcal{F}^{rad}(\mathcal{M}) \to \mathcal{F}(\mathscr{I}^+)$

Bulk to Boundary correspondence III

- Define a **state on the boundary** $\omega_{\mathscr{I}} : \mathcal{F}(\mathscr{I}^+) \to \mathbb{C}$.
- Pull $\omega_{\mathscr{I}}$ **back to the bulk** via ι to get a state $\omega_{\mathcal{M}}$ on $\mathcal{F}^{rad}(\mathcal{M})$:

$$\omega_{\mathcal{M}} \doteq \iota^* \omega_{\mathscr{I}} = \omega_{\mathscr{I}} \circ \iota\,.$$

- **Distinguished choice**: *Invariance under the BMS group*.
- Via pull-back, we get the state on the bulk.
- This state turns out to be:
  - of ***Hadamard form,*** [*C. Gérard, M. Wrochna: Construction of Hadamard states by characteristic Cauchy problem.*]
  - ***invariant under the action of all isometries of the bulk.*** [*Moretti : Uniqueness theorem for BMS-invariant states of scalar QFT on the null boundary of asymptotically flat spacetimes and bulk-boundary observable algebra correspondence*]

Conclusions: THANK YOU for your attention!

Asymptotic analysis is

- a powerful tool to construct physically relevant states
- it works for all free fields with some problems for linearized gravity

What comes next?

- Prove if the no-go result for the GX gauge cannot be circumvented with another gauge
- Apply this method for specific more complicated black hole backgrounds